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ANALYSIS AND TESTS OF RIGIDLY CONNECTED REINFORCED CONCRETE FRAMES

BY

MIKISHI ABE

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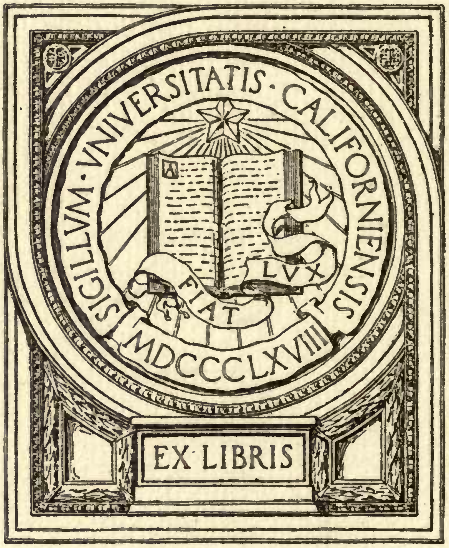

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ANALYSIS AND TESTS OF RIGIDLY CONNECTED REINFORCED CONCRETE FRAMES

BY
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CONTENTS

I. INTRODUCTION

PAGE
1. Preliminary 9
2. The Use and the Advantages of the Rigidly Connected Reinforced Concrete Frame 10
3. Scope of Investigation and Acknowledgment 12

II. THE ANALYSIS OF RIGIDLY CONNECTED FRAMES

4. Notation 14
5. Statically Determinate and Indeterminate Systems and Number of Statically Indeterminates 15
6. Principle of Least Work 18
7. The Effect of Work of Normal Force on the Magnitude of Statically Indeterminate Forces 20
8. Simple Frames under Vertical Load 25
9. Single Story Construction with Three Spans 30
10. Trestle Bent with Tie 34
11. Building Construction with Several Stories and Spans . . 35
12. Frame with Three Spans 39
13. Square Frame under Horizontal Load 44
14. The Nature of the Resulting Formulas; Relation between Horizontal Reactions in Frame under Uniform Load and under Concentrated Load 45
15. Effect of Variation in Moment of Inertia and Relative Height of Frame on Bending Moment in Horizontal Member 48
16. Effect of Variation in Moment of Inertia on Bending Moment in Vertical Member 52

399192

III. TESTS OF RIGIDLY CONNECTED REINFORCED CONCRETE FRAMES

PAGE
17. Test Specimens 55
18. Materials 59
19. Making and Storage of Test Frames 60
20. Testing 61
21. Explanation of Tables and Diagrams 70
22. Phenomena of Frame Tests 71
23. Conditions Considered in the Comparison of Tests with Analysis 91
24. Comparison of Test Results with Analyses 92
25. Effect of End Condition of Column on Results 96
26. Distribution of Stress over the Cross-Section 96
27. Position of Point of Inflection in Columns 98
28. Continuity of the Composing Members of a Frame . . . 98
29. Stresses at Corners 99
30. Conclusions and General Comments 100

LIST OF FIGURES

NO. PAGE

1. Simple Hinged Frame with Concentrated Load 15
2. Illustrations of Degree of Indeterminateness by Means of Joint Bars . . 16
3. Types of Frames of Various Degrees of Indeterminateness 17
4. Type of Structure in which Effect of Normal Force is Great 18
5. Unsymmetrical Frame under Concentrated Load 20
6. Simple Frames under Vertical Load; Lower Ends of Columns Hinged . 24
7. Simple Frames under Vertical Load; Lower Ends of Columns Fixed . . 28
8. Rectangular Frame with Rigidly Connected Tie at Base of Columns . . 28
9. L-frame with Column Hinged at Support 29
10. L-frame with Column Fixed at Support 30
11. Continuous Single Story Spans; One Panel Loaded 31
12. Single Story Three-Span Frame, Having Middle Span Uniformly Loaded . 31
13. Single Story Three-Span Frame under Concentrated Load 34
14. A-frame Showing Form of Moment Curve 35
15. Cross-Section of Building Frame with Single Panel Loaded 36
16. Two-Story Three-Span Frame Having Middle Span Uniformly Loaded . 36
17. Coefficient of Bending Moments for Top of Lower Column and Foot of Upper Column for Two-Story Three-Span Frame Having All External Connections Fixed 38
18. Formulas for Single Story Three-Span Frame with Four Columns Having Columns Hinged at Lower End 40
19. Formulas for Single Story Three-Span Frame with Four Columns Having Columns Fixed at Lower End 41
20. Single Story Three-Span Frame with Four Columns Having Middle Span Uniformly Loaded 42
21. Formulas for Rectangular Frame under Horizontal Load and for Octagonal and Rectangular Reservoirs 43
22. Horizontal Reactions for Distributed Loads and Concentrated Loads . 44
23. Locus of Intersection of Reaction Lines in Single Span Single Story Frame under Single Concentrated Load 47
24. Coefficient of Bending Moment at Center of Middle Span for Single Story Three-Span Frame Having Extreme Ends of Columns and Beams Hinged 49
25. Coefficient of Bending Moment at Center of Middle Span for Single Story Three-Span Frame Having Extreme Ends of Beams Hinged and Column Ends Fixed 50
26. Coefficient of Bending Moment at Center of Middle Span for Single Story Three-Span Frame Having All External Connections Fixed . . 51

NO. PAGE
27. Coefficient of Bending Moment at Center of Middle Span for Single Story Three-Span Frame Having Lower Ends of Columns Hinged . . 52
28. Coefficient of Bending Moment at Center of Middle Span for Single Story Three-Span Frame Having Lower Ends of Columns Fixed . . . 53
29. Coefficient of Bending Moment at Lower Ends of Interior Columns for Single Story Three-Span Frame Having Lower Ends of Columns Fixed 54
30. Size and Distribution of Reinforcing Bars in Frames 1 and 6 55
31. Size and Distribution of Reinforcing Bars in Frames 2 and 4 56
32. Size and Distribution of Reinforcing Bars in Frames 3 and 7 56
33. Size and Distribution of Reinforcing Bars in Frame 5 57
34. Size and Distribution of Reinforcing Bars in Frame 8 57
35. Stress-Deformation Diagrams for Cylinders 61
36. Observed and Computed Stresses in Frame 1 62
37. Observed and Computed Stresses in Frame 2 63
38. Observed and Computed Stresses in Frame 3 64
39. Observed and Computed Stresses in Frame 4 65
40. Observed and Computed Stresses in Frame 5 66
41. Observed and Computed Stresses in Frame 6 67
42. Observed and Computed Stresses in Frame 7 68
43. Observed and Computed Stresses in Frame 8 69
44. Points Used for Comparison of Computed and Observed Stresses . . . 76
45. Load-Deformation Diagrams for Additional Gage Lines, Frames 1, 2, 3 78
46. Location of Additional Gage Lines, Frames 1, 2, 3 79
47. Load-Deformation Diagrams for Additional Gage Lines, Frames 4, 5, 6, 7 . 80
48. Location of Additional Gage Lines, Frames 4, 5, 6, 7 81
49. View of Frame 1 in Testing Machine 83
50. View of Frame 5 after Test 83
51. View of Frame 1 after Test 84
52. View of Frame 2 after Test 84
53. View of Frame 3 after Test 85
54. View of Frame 4 after Test 85
55. View of Frame 6 after Test 86
56. View of Frame 7 after Test 86
57. View of Frame 8 after Test 89
58. Connection of Beam and Column Showing Typical Location of First Cracks 98
59. Rectangular Joint with and without Fillet 99

LIST OF TABLES

NO. PAGE

1. Elements Used in Conditional Equations for Solution of an Indeterminate Structure 21
2. Elements Used in Conditional Equations for Solution of Inverted U-frame 26
3. Formulas for Reactions, Bending Moments, and Points of Inflection for Single Story Three-Span Frames 32
4. Coefficients of Bending Moment for Single Story Three-Span Frame . . 33
5. Formulas for Two-Story Three-Span Frame 37
6. Coefficients of Bending Moment for Two-Story Three-Span Frame with Ends of Columns and Beams Fixed 39
7. Coefficients of Bending Moment for Single Story Three-Span Frame Having Four Columns 42
8. Data of Test Frames 58
9. Tension Tests of Reinforcing Steel 59
10. Compression Tests of Concrete Cubes and Cylinders 60
11. Values of Modulus of Elasticity of Concrete Used in Stress Computations 71
12. General Data of Tests of Frames 72, 73
13. Computed Stresses for Test Frames in Terms of Total Applied Load . 74, 75
14. Computed and Observed Stresses in Frames 94, 95
15. Stresses in Bars at Front and Back of Members of Test Frames . . . 97
16. Observed Stresses at Sharp Corners in Frames 1, 3 and 6 100

and test have been made. The general rigidity at the places where the members are joined has been investigated. The work is presented in the thought that it will be helpful in bringing into wider use the principles applicable to the design of rigidly connected reinforced concrete constructions.

2. *The Use and the Advantages of the Rigidly Connected Reinforced Concrete Frame.*—Since about 1905 reinforced concrete frame construction has been extensively used in continental Europe. Many examples can be found in the German texts and magazines. In England also frame constructions of reinforced concrete viaducts and other structures have been built in recent years. There is also a tendency in America to use reinforced concrete frames for buildings and bridges.

The field of the application of rigid frames is almost unlimited, for most reinforced concrete structures are composed of elements of rigid frames. It covers such constructions as buildings, bridge structures, trestles and viaducts, culverts and sewers, subway construction, retaining walls, and reservoirs and water tanks. In these structures rigid connections are used between members and in many or most of them the bending moments are statically indeterminate.

It is clear that every building construction of reinforced concrete may be considered as a rigidly connected frame, for columns, girders, beams and slabs are all rigidly connected with each other, even though the effect of this condition is not fully considered in the design. In continental European countries it is most common to use frames in building constructions, such as roofs, balconies, towers, and the building as a whole. In the design the requirements and the advantages of the frame are taken into account.

Bridge structures are in the field of the rigid frame. Arches, beam and bent construction, and most bridge structures can be designed as frames on a rigid analytical basis. In highway bridges, for example, a spandrel-braced arch is frequently used. In such a case columns are rigidly connected to the arch ribs and to the superstructure, and therefore the design should be made as a rigidly connected frame. The designing of trestles and viaducts as a frame will secure safety and at the same time obtain the best proportioning of parts.

Box culverts and the box type of construction for subways give sections which are examples of the rigidly connected frame and which may not be rationally designed without a sufficient knowledge of rigid frames.

In water tanks and reservoirs of a rectangular or a polygonal form, the unknown negative bending moment due to a rigid connection of wall to wall or base to wall will exist at each corner. These moments are modified by the relative thickness of walls and the other dimensions of the structure. A knowledge of the rigid frame will suggest the proper method of solution.

It can thus be seen that most monolithic construction falls within the field of the rigid frame. A study of the rigid frame will assist in developing judgment for use in the design of such construction.

In building and structural design, insufficient attention is often given to the bending of columns caused by the rigidity of connections. The bending moment for a beam is frequently taken as an assumed fraction of Pl (where P is the load and l is the span) while bending moments at the ends and in the columns are disregarded entirely, thus leaving the structure inadequate or making one part stronger at the expense of the other.

The reinforced concrete frame is advantageous in that material can be saved and a much better result obtained from the theoretical and structural point of view. In ordinary concrete building construction the element of rigidity is usually not fully taken advantage of. With the concrete frame construction, however, the rigidity of the connection of the members may be used. The rigid frame is capable of exact design, and therefore the economical distribution of materials can be realized.

One reason why some engineers hesitate to use concrete frames extensively is that they hardly believe in the continuity of the parts of the structure and doubt the effect of the rigidity of the connection. The question is also naturally raised if the formulas deduced from the elastic work of deformation of a non-homogeneous material like reinforced concrete will hold good for such composite members with fair agreement; furthermore the secondary stresses may act to modify the results. Under actual conditions, as is well known, the fundamental assumptions which underlie the static considerations can seldom be more than partially fulfilled even under carefully prepared specifications and well executed designs. These things must be considered before coming to a conclusion as to the reliability of rigidly connected reinforced concrete frames. It is evident that reinforced concrete frames will be reliable if there is perfect continuity or complete rigidity of joint, close agreement between theory and experiment, and a small effect of stresses of a secondary character. It is not known that an experimental study of this subject has before been made. It may be expected then that careful experiments and investigation will give information which will help to settle these questions.

3. *Scope of Investigation and Acknowledgment.*—In this bulletin formulas for several types of statically indeterminate structures which have been deduced by the use of the principle of least work are given. For vertical load the following cases have been analyzed: (1) single story, single span; (2) single story, three spans; (3) trestle bent with tie, single span; (4) building frame with several stories and several spans; and (5) bridge trestle. For horizontal load the following cases have been analyzed: (1) single story, single span; (2) octagonal reservoir or tank; and (3) rectangular reservoir or tank.

In order to put to practical test the reliability of these formulas for reinforced concrete structures, eight test frames designed according to the formulas found by the analyses were made, and the deformations produced in the various parts of the members by the series of test loads were measured. In the design of the frames requirements not touched upon by the analyses referred to were provided for in a practical way. The analyses and the results of the tests have been subjected to critical study and discussion. The specimens were made in November and December, 1913, and January, 1914, and were tested in January, February, and March, 1914. In making these tests the purpose was to obtain experimental information along the following lines which have a bearing on the design of rigidly connected reinforced concrete frames:

(1) The amount and the distribution of stresses in the reinforcement and in the concrete
(2) The continuity of the composing members of a frame
(3) The location of sections of critical stress
(4) The reliability of a reinforced concrete frame
(5) The applicability of the theoretical formulas in the design of frames.

The experimental work was done as a research problem of the Engineering Experiment Station of the University of Illinois.

The work was under the charge of PROFESSOR ARTHUR N. TALBOT, to whom, as well as to other members of the staff, acknowledgment is due for valuable suggestions and aid.

The limits of space set for this bulletin will not permit publication of even a small part of the details of the derivation of the formulas found nor of the observations, calculations, and other data of the tests. Instead the plan has been followed generally of giving the formulas found from the analyses without the details of the derivation and, in the case of the experimental work, of showing graphically the main stresses that were observed at the principal loads, and of not including details of the data. The original and the reduced data and more detailed work of the analyses are on file at the Laboratory of Applied Mechanics of the University of Illinois.

II. THE ANALYSIS OF RIGIDLY CONNECTED FRAMES

4. *Notation.*—The following notation is used generally throughout the bulletin:

A = area of cross-section of a member. Numerical suffixes are used for individual members when a frame is composed of members of different sizes.
A is also used as a coefficient to represent certain algebraic expressions.
a = distance from the left corner or axis of a frame to the point of application of a concentrated load on a top beam.
b = distance from the right corner or axis of a frame to the point of application of a concentrated load on a top beam.
E = modulus of elasticity (considered as constant) of the material.
H = horizontal reaction acting at the end of a column.
I = moment of inertia in general.
h = total vertical height of frame.
l = total length of horizontal span of frame.
s = length of an inclined member.
m = ratio of moment of inertia of horizontal member to that of vertical member.
$n = \frac{h}{l}$ = ratio of height of frame to length of span.
M = bending moment in general.
N = normal force or stress on a section (total internal force normal to the section).
P = a concentrated load.
p = intensity of a uniformly distributed load.
V = vertical reaction.
f_s = unit stress in steel in tension.
f'_s = unit stress in steel in compression.
f_c = unit stress in concrete in compression.

In the diagrams representing the forms of the frames analyzed, the ends of members are indicated by lower case letters and the symbols for the properties, forces, and moments use these letters as subscripts to indicate the members to which they apply as well as the point of application. The method of use will be made clear by the following examples and by reference to Fig. 6:

H_a = horizontal reaction at a.
I_{ab} = moment of inertia of member ab.
h_{bc} = vertical height of bc.
l_{bc} = length of horizontal projection of bc.
M_{bc} = bending moment at any point in bc.

5. *Statically Determinate and Indeterminate Systems and Number of Statically Indeterminates.*—A force is said to be statically determinate when its direction and magnitude and its point of application are known from the conditions of static equilibrium. The conditions of static equilibrium for any number of forces in a plane, as is generally well known, are three, that is to say,

(1) $Y=0$, or the algebraic sum of all vertical forces acting on a body is equal to zero.
(2) $X=0$, or the algebraic sum of all horizontal forces acting on a body is equal to zero.
(3) $M=0$, or the algebraic sum of the moments of all forces is equal to zero.

The loads to which structures may be subjected are always given. The other external forces are the reactions due to the loads. The reactions are exerted by the supports of the structure, and in order that they may be determined from the statical conditions the total number of unknowns must not exceed three. The ordinary trusses without redundant members are always statically determinate if a frictionless pin is used at each joint and if in the determination the effect of the longitudinal deformation of members on the stresses is neglected. If a case in which two members meet at a joint is considered as shown in Fig. 1, two unknown forces exist at the joint, the vertical

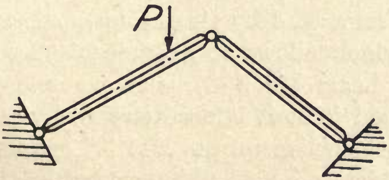

FIG. 1. SIMPLE HINGED FRAME WITH CONCENTRATED LOAD

and horizontal forces, and therefore the total number of unknown forces due to the external force P is six. But each member will give three statical conditions as stated before, and therefore this is a statically determinate system.

In studying the behavior of statically determinate and statically indeterminate systems, the conception of the connection of members by means of joint bars is a convenience. In Fig. 2 (a) the two members are not connected. They are entirely free to move horizontally and vertically and also are free in rotation; accordingly it may be called the arrangement having three freedoms in motion. An example of this arrangement is a touching joint between the free ends of cantilever beams as shown in Fig. 2 (b). In Fig. 2 (c) two members are connected by a single bar, and they are prevented from moving vertically, but are free to move horizontally and to rotate about a point A. This may be called the arrangement having two freedoms in motion. An example of this arrangement is the frictionless roller end of a cantilever as shown in Fig. 2 (d). In Fig. 2 (e) two members are connected by two connecting bars and have only one freedom in motion, that is, the rotation of a member about A, the intersecting point of two bars. The crown hinge of an arch, Fig. 2 (f), is an example. In Fig. 2 (g), two members are connected by three joint-bars, and may be called a rigid connection which allows no freedom of motion. The restrained end of a cantilever beam, Fig. 2 (h), is an example of this arrangement.

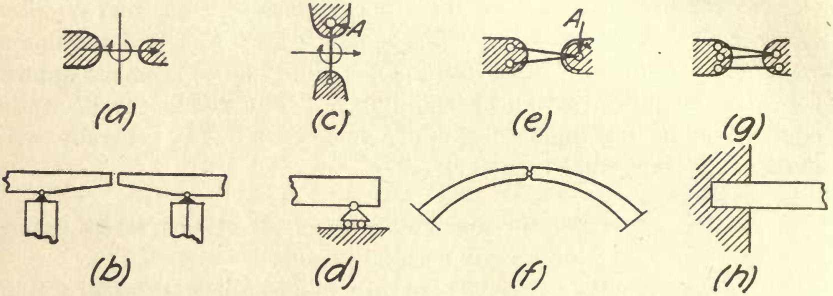

FIG. 2. ILLUSTRATIONS OF DEGREE OF INDETERMINATENESS BY MEANS OF JOINT BARS

To make a rigid joint it is necessary to have three joint bars at each connection between members, and $3S$ conditions of equilibrium must be set up to determine $3S$ unknowns when the structure is composed of S members. If the structure is rigidly connected to the ground, more conditions than $3S$ are required. Let a be the number of joint-bars needed to connect one member to another, and b be the number of joint bars needed to connect the member to the ground; then from the existing $3S$ conditions the following relation is necessary to make the structure statically determinate,

$$a+b=3S$$

When the members in the structure are all rigidly connected to each other, $a+b$ always exceeds $3S$ and therefore the case becomes a statically indeterminate system in which

$$a+b-3S=m$$

where m represents the number of the statically indeterminate forces. Such a system is called m-fold statically indeterminate, and m additional equations of condition are necessary to determine these unknowns.

Fig. 3 gives a few examples.

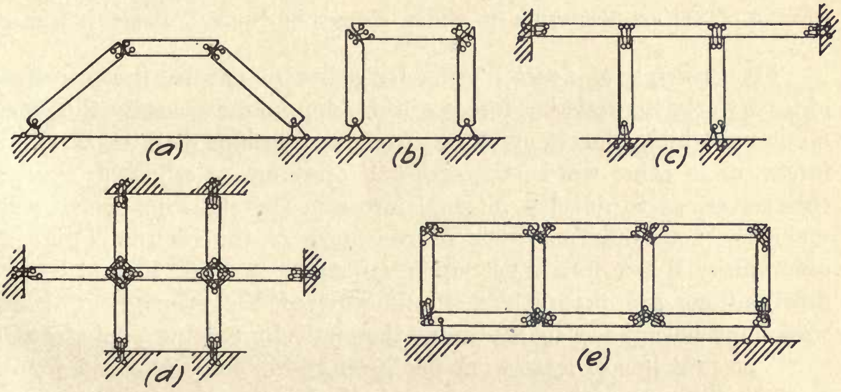

FIG. 3. TYPES OF FRAMES OF VARIOUS DEGREES OF INDETERMINATENESS

Case a. $a+b-3S=5+4-9=0$, therefore statically determinate.

Case b. $a+b-3S=6+4-9=1$, 1-fold statically indeterminate.

Case c. $a+b-3S=6+12-9=9$, 9-fold statically indeterminate.

Case d. $a+b-3S=18+18-21=15$, 15-fold statically indeterminate.

Case e. $a+b-3S=36+3-30=9$, 9-fold statically indeterminate.

In general the change in length of a member due to the direct stresses in the member will have an effect on the magnitude of the stresses developed. However, as is shown in a later paragraph, this effect is very slight in all ordinary forms of construction and has been neglected in stating the method of determining the degree of indeterminateness. Consequently in any member in which the change in length due to the direct stress has an appreciable effect on the stress developed, the criterion stated does not apply. Fig. 4 illustrates such a case.

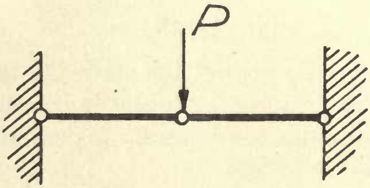

FIG. 4. TYPE OF STRUCTURE IN WHICH EFFECT OF NORMAL FORCE IS GREAT

6. *Principle of Least Work.*—By a law of nature, the principle of least work, the resisting forces will develop no more energy than the minimum which is necessary to maintain equilibrium with the external forces; or in other words, the external forces are so adjusted, among themselves, as to develop internal forces in the structure which will make the total internal work of resistance of the internal forces a minimum. When forces act upon an elastic system in which the deformations are proportional to the stresses, the principle of least work may be applied to determine the statically indeterminate forces.

The principle of least work has been known for a hundred years, but the first complete announcement of this theorem was given by Castigliano in 1879. Professor Cain expresses the principle in the following words:*

"The elastic forces experienced between the molecules after deformation correspond to a minimum of the work of deformation of the system, expressed as a function of certain stresses, taken with respect to these stresses successively, regarded as independent during the differentiation."

Professor Hiroi† translates Castigliano's expression of the principle of least work in the following words: "The partial derivatives of the work of resistance with respect to statically indeterminate forces which are so chosen that the forces themselves perform no work are equal to zero.''

---

*See Trans. Amer. Soc. Civ. Eng., Vol. XXIV, p. 291.

†See Statically Indeterminate Stresses, by I. Hiroi.

The total internal work may be subdivided into the parts due to bending moment, normal stress, and shearing stress.

The total work due to the bending moment M in a member will be

$$w_1=\int\frac{M^2dx}{2EI}$$

For the total internal work due to a total normal stress* N on a section

$$w_2=\int\frac{N^2dx}{2EA}$$

If shearing stress S is uniform over the cross-section, the expression for the internal work due to shearing stress is

$$w=\int\frac{S^2dx}{2GA}$$

where G expresses the shearing modulus of elasticity of a material. Since, however, the shearing stress is not uniform over the cross-sections, the expression for the internal work due to shear is modified, and

$$w_3=\int\frac{KS^2dx}{2GA}$$

where K is a known factor for a specified form of the cross-section. Therefore for the total work of resistance,

$$W=w_1+w_2+w_3=\frac{1}{2}\int\frac{M^2dx}{EI}+\frac{1}{2}\int\frac{N^2dx}{EA}+\frac{1}{2}\int\frac{KS^2dx}{GA}$$

Suppose that there are n statically indeterminate forces P_1, P_2, ... P_n in an elastic system. According to the theorem of Castigliano

$$\frac{\partial W}{\partial P_1}=\int\frac{M}{EI}\,\frac{\partial M}{\partial P_1}dx+\int\frac{N}{EA}\,\frac{\partial N}{\partial P_1}dx+\int\frac{KS}{GA}\,\frac{\partial S}{\partial P_1}dx=0$$

$$\frac{\partial W}{\partial P_2}=\int\frac{M}{EI}\,\frac{\partial M}{\partial P_2}dx+\int\frac{N}{EA}\,\frac{\partial N}{\partial P_2}dx+\int\frac{KS}{GA}\,\frac{\partial S}{\partial P_2}dx=0$$

. . . .

$$\frac{\partial W}{\partial P_n}=\int\frac{M}{EI}\,\frac{\partial M}{\partial P_n}dx+\int\frac{N}{EA}\,\frac{\partial N}{\partial P_n}dx+\int\frac{KS}{GA}\,\frac{\partial S}{\partial P_n}dx=0$$

---

*Normal to a cross-section of the member.

These furnish as many equations of condition as there are unknown quantities. The solution of these equations will give the exact formulas for statically indeterminate quantities.

7. *The Effect of Work of Normal Force on the Magnitude of Statically Indeterminate Forces.*—In the general equation

$$\int \frac{M}{EI}\,\frac{\partial M}{\partial P}dx + \int \frac{N}{EA}\,\frac{\partial N}{\partial P}dx + \int \frac{KS}{GA}\,\frac{\partial S}{\partial P}dx = 0$$

the second term will disappear when $\frac{\partial N}{\partial P}$ is equal to zero or, in other words, when the normal force* does not contain any of the statically indeterminate quantities.

When a frame is fixed at its column ends, the vertical and horizontal reactions and the bending moment at the fixed column ends are statically indeterminate. Generally the normal force in any member contains these reactions as factors. But if a frame having a single span is symmetrical in form and in the manner of loading, the vertical reactions become statically determinate, and therefore the horizontal reaction is the only indeterminate term which enters into the expression of the normal force. If, at the same time, the columns are vertical, the horizontal reactions do not affect the normal forces in the columns, and in the horizontal members only are the normal forces affected by a statically indeterminate force, namely, the horizontal reaction.

From these statements it will be seen that the form of frame which will be largely affected by the normal force is that having a sloped column under a vertical load.

The frame shown in Fig. 5 is used to illustrate the method of

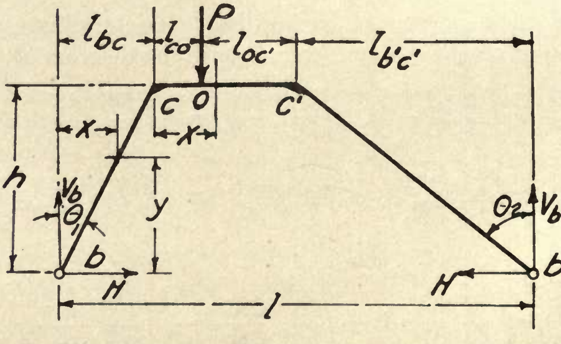

FIG. 5. UNSYMMETRICAL FRAME UNDER CONCENTRATED LOAD

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*Normal to a cross-section of the member.

TABLE 1

ELEMENTS USED IN CONDITIONAL EQUATIONS FOR SOLUTION OF AN INDETERMINATE STRUCTURE

Member	Moment of Inertia	Limits of Integration	Bending Moment in the Member	Normal Forces in the Member	$\frac{\partial M}{\partial H}$	$\frac{\partial N}{\partial H}$
bc	I_{bc}	From zero to h	$V_b x - Hy = V_b y \tan\theta_1 - Hy$	$V_b \cos\theta_1 + H \sin\theta_1$	$-y$	$\sin\theta_1$
co	$I_{cc'}$	From zero to l_{co}	$V_b(l_{bc} + x) - Hh$	H	$-h$	$+1$
oc'	$I_{cc'}$	From l_{co} to $l_{oc'}$	$V_b(l_{bc} + x) - Hh - P(x - l_{co})$	H	$-h$	$+1$
$b'c'$	$I_{b'c'}$	From zero to h	$V_{b'} x - Hy = V_{b'} y \tan\theta_2 - Hy$	$V_{b'} \cos\theta_2 + H \sin\theta_2$	$-y$	$\sin\theta_2$

analysis and to bring out the effect of the work of the direct normal force on the magnitude of the statically indeterminate forces. In this case H is the only statically indeterminate force.

Taking the moment of all forces about b'

$$V_b\, l - P(l_{oc'} + l_{b'c'}) = 0 \text{ or } V_b = \frac{l_{b'c'} + l_{oc'}}{l} P = KP$$

and

$$V_{b'} = P - KP \text{ or } V_{b'} = (1-K)P$$

K being used as a general coefficient.

In general the internal work due to shearing stress may be neglected, and the general equation becomes

$$\int \frac{M}{EI} \frac{\partial M}{\partial H} dx + \int \frac{N}{EA} \frac{\partial N}{\partial H} dx = 0$$

All necessary elements in forming this equation are arranged in Table 1.

Inserting these values in the general equation gives the following expression, in which it is assumed that E and I are constant:

$$\frac{1}{EI_{bc}} \int_0^h (V_b \tan\theta_1 y - Hy)(-y) \sec\theta_1 dy + \frac{1}{EI_{cc'}} \int_0^{l_{co}} (V_b l_{bc} + V_b x - Hh)(-h) dx$$

$$+ \frac{1}{EI_{cc'}} \int_{l_{co}}^{l_{cc'}} \left\{ V_b l_{bc} + V_b x - Hh - P(x - l_{co}) \right\} (-h) dx$$

$$+ \frac{1}{EI_{b'c'}} \int_0^h (V_{b'} \tan\theta_2 y - Hy)(-y) \sec\theta_2 dy$$

$$+ \frac{1}{EA_{bc}} \int_0^h (V_b \cos\theta_1 + H \sin\theta_1) \sin\theta_1 \sec\theta_1 dy + \frac{1}{EA_{cc'}} \left\{ \int_0^{l_{co}} H dx + \int_{l_{co}}^{l_{cc'}} H dx \right\}$$

$$+ \frac{1}{EA_{b'c'}} \int_0^h \left\{ V_{b'} \cos\theta_2 + H \sin\theta_2 \right\} \sin\theta_2 \sec\theta_2 dy = 0$$

Integrating and simplifying this equation gives the following general expression for the statically indeterminate force H, in which the work of the normal force is fully counted:

$$= \frac{K\left[\frac{l_{bc}S_{bc}}{3I_{bc}}+\frac{2l_{bc}l_{cc'}+l^2_{cc'}}{2I_{cc'}}-\frac{l_{b'c'}S_{b'c'}}{3I_{b'c'}}-\frac{\sin\theta_1}{A_{bc}}+\frac{\sin\theta_2}{A_{b'c'}}\right]-\left[\frac{(l_{cc'}-l_{oc})^2}{2I_{cc'}}-\frac{S_{b'c'}l_{b'c'}}{3I_{b'c'}}+\frac{\sin\theta_2}{A_{b'c'}}\right]}{h\left[\frac{S_{bc}}{3I_{bc}}+\frac{l_{cc'}}{I_{cc'}}+\frac{S_{b'c'}}{3I_{b'c'}}\right]+\left[\frac{\sin\theta_1\tan\theta_1}{A_{bc}}+\frac{\sin\theta_2\tan\theta_2}{A_{b'c'}}+\frac{l_{oc'}}{hA_{cc'}}\right]}P$$

where $K = \frac{l_{b'c'}+l_{oc'}}{l}$

In this formula, the terms which contain A_{bc}, $A_{cc'}$, and $A_{b'c'}$ enter because of taking into consideration the work of the normal forces represented by the second term of the general equation

$$\int\frac{M}{EI}\frac{\partial M}{\partial H}dx+\int\frac{N}{EA}\frac{\partial N}{\partial H}dx=0$$

If the effect of the normal force is neglected, then

$$H = \frac{K\left[\frac{l_{bc}S_{bc}}{3I_{bc}}+\frac{2l_{bc}l_{cc'}+l^2_{cc'}}{2I_{cc'}}-\frac{l_{b'c'}S_{b'c'}}{3I_{b'c'}}\right]-\left[\frac{(l_{cc'}-l_{oc})^2}{2I_{cc'}}-\frac{S_{b'c'}l_{b'c'}}{3I_{b'c'}}\right]}{h\left[\frac{S_{bc}}{3I_{bc}}+\frac{l_{cc'}}{I_{cc'}}+\frac{S_{b'c'}}{3I_{b'c'}}\right]}P$$

In most cases a value of θ greater than 30 degrees will not be used, because an increase in θ rapidly increases the horizontal reaction at the end of the column.

Assume a case in which $l_{bc}=l_{cc'}=l_{b'o'}=h=120$ inches. $\theta_1=\theta_2=$ 45 degrees, $A_{bo}=A_{cc'}=A_{b'c'}=10$ by 12 inches. $I_{bc}=I_{cc'}=I_{b'c'}=1{,}000$ in.^{4} $K=\frac{1}{2}$

When the effect of the work of the direct force is considered,

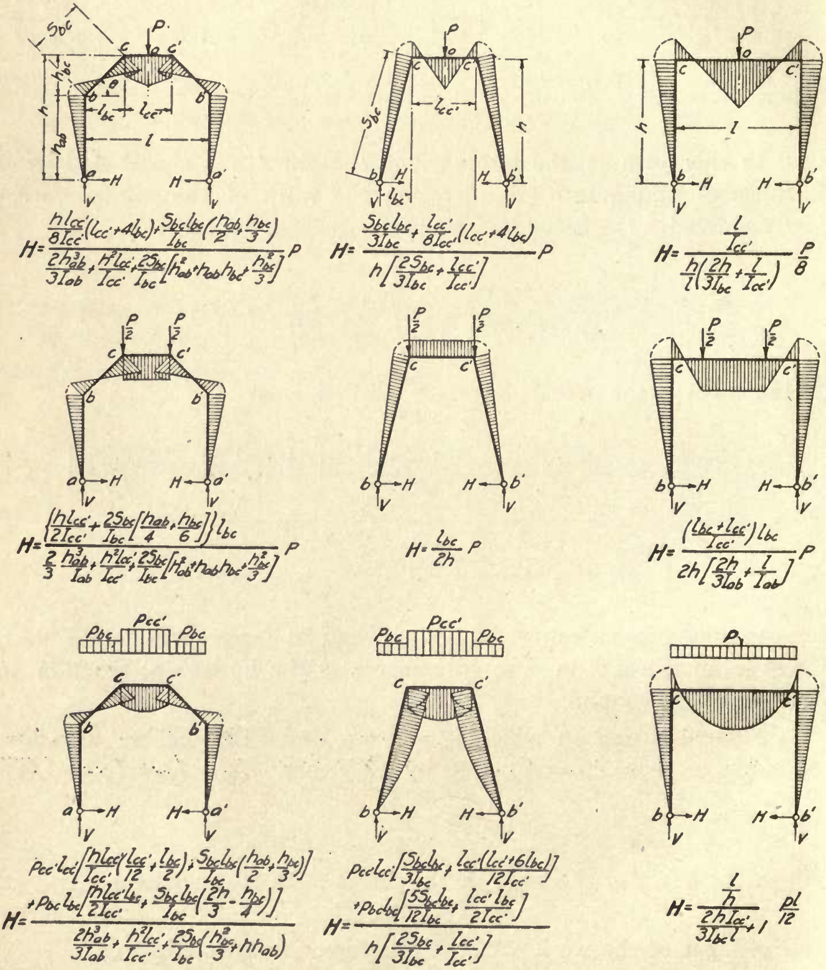
$$H=0.5637P$$

If the effect of the work of the direct force is neglected,

$$H=0.5643P$$

The difference $0.0006P$ ($0.0011H$) is inconsiderable.

In the foregoing example it is seen that the final formula is very much complicated by taking the direct force into consideration and that the effect of the internal work of all the direct stresses on the final value for statically indeterminate stresses is inconsiderable when compared with that of the bending moment. The work of the normal force is therefore disregarded in making the analyses of the frames treated in this bulletin.



$$H=\frac{\frac{hl_{cc'}}{8I_{cc'}}(l_{cc'}+4l_{bc})+\frac{S_{bc}l_{bc}}{I_{bc}}\left(\frac{h_{ab}}{2}+\frac{h_{bc}}{3}\right)}{\frac{2h_{ab}^3}{3I_{ab}}+\frac{h^2l_{cc'}}{I_{cc'}}+\frac{2S_{bc}}{I_{bc}}\left[h_{ab}^2+h_{ab}h_{bc}+\frac{h_{bc}^2}{3}\right]}P$$

$$H=\frac{\frac{S_{bc}l_{bc}}{3I_{bc}}+\frac{l_{cc'}}{8I_{cc'}}(l_{cc'}+4l_{bc})}{h\left[\frac{2S_{bc}}{3I_{bc}}+\frac{l_{cc'}}{I_{cc'}}\right]}P$$

$$H=\frac{\frac{l}{I_{cc'}}}{\frac{h}{l}\left(\frac{2h}{3I_{bc}}+\frac{l}{I_{cc'}}\right)}\frac{P}{8}$$

$$H=\frac{\left\{\frac{hl_{cc'}}{2I_{cc'}}+\frac{2S_{bc}}{I_{bc}}\left[\frac{h_{ab}}{4}+\frac{h_{bc}}{6}\right]\right\}l_{bc}}{\frac{2}{3}\frac{h_{ab}^3}{I_{ab}}+\frac{h^2l_{cc'}}{I_{cc'}}+\frac{2S_{bc}}{I_{bc}}\left[h_{ab}^2+h_{ab}h_{bc}+\frac{h_{bc}^2}{3}\right]}P$$

$$H=\frac{l_{bc}}{2h}P$$

$$H=\frac{\left(\frac{l_{bc}+l_{cc'}}{I_{cc'}}\right)l_{bc}}{2h\left[\frac{2h}{3I_{ab}}+\frac{l}{I_{ab}}\right]}P$$

$$H=\frac{p_{cc'}l_{cc'}\left[\frac{hl_{cc'}}{I_{cc'}}\left(\frac{l_{cc'}}{12}+\frac{l_{bc}}{2}\right)+\frac{S_{bc}l_{bc}}{I_{bc}}\left(\frac{h_{ab}}{2}+\frac{h_{bc}}{3}\right)\right]+p_{bc}l_{bc}\left[\frac{hl_{cc'}l_{bc}}{2I_{cc'}}+\frac{S_{bc}l_{bc}}{I_{bc}}\left(\frac{2h}{3}-\frac{h_{bc}}{4}\right)\right]}{\frac{2h_{ab}^3}{3I_{ab}}+\frac{h^2l_{cc'}}{I_{cc'}}+\frac{2S_{bc}}{I_{bc}}\left(\frac{h_{bc}^2}{3}+hh_{ab}\right)}$$

$$H=\frac{p_{cc'}l_{cc'}\left[\frac{S_{bc}l_{bc}}{3I_{bc}}+\frac{l_{cc'}(l_{cc'}+6l_{bc})}{12I_{cc'}}\right]+p_{bc}l_{bc}\left[\frac{5S_{bc}l_{bc}}{12I_{bc}}+\frac{l_{cc'}l_{bc}}{2I_{cc'}}\right]}{h\left[\frac{2S_{bc}}{3I_{bc}}+\frac{l_{cc'}}{I_{cc'}}\right]}$$

$$H=\frac{\frac{l}{h}}{\frac{2hI_{cc'}}{3I_{bc}l}+1}\frac{pl}{12}$$

FIG. 6. SIMPLE FRAMES UNDER VERTICAL LOAD; LOWER ENDS OF COLUMNS HINGED

Attention is called to the fact that even though the effect of the internal work of the direct stresses may be neglected in determining the reactions the direct stresses themselves can not be neglected when calculating the total stress in any member. The direct stresses may be added algebraically after the statically indeterminate stresses are found.

The effect of the work of the deformation due to shear is generally so insignificant when compared with that due to the bending that it may be entirely neglected without sensible error in the calculation of the internal work.

8. *Simple Frames under Vertical Load.*—In Fig. 6 are given forms of a type of simple frame. The form at the left of the figure is the general form of the inverted U-frame, and the others are special cases of this frame. The column connections at the base are hinged and all other joints are rigid. Three forms of vertical loading are presented—a single concentrated load, two concentrated loads, and a uniform load. For this frame the statically indeterminate force is the horizontal reaction H. Formulas for H, derived by analysis using the principle of least work, are given in the figure for the three forms of frame and for the three loadings.

Knowing the horizontal reaction H, the bending moment and the forces at any section of the frame may be determined by the ordinary analytical method. For example, the bending moment at the middle of the top beam for the frames with two concentrated loads shown in Fig. 6 is equal to the algebraic sum of the moment of the horizontal reaction H with a moment arm equal to the height of the frame, the moment of the vertical reaction V about the section considered, and the moment of a load $\frac{P}{2}$ about the section.

As a specimen application of the method of using the principle of least work, the general solution of the frame and loading shown in the upper left-hand corner of Fig. 6 is given. The statically indeterminate force is H. The effect of normal forces being neglected, the general equation of condition is

$$\int \frac{M}{EI} \frac{\partial M}{\partial H} dx = 0$$

All quantities necessary in forming the conditional equation for this case are arranged in Table 2.

TABLE 2

ELEMENTS USED IN CONDITIONAL EQUATIONS FOR SOLUTION OF INVERTED U-FRAME

Member		I	M	$\frac{\partial M}{\partial H}$
ab	*a'b'*	I_{ab}	$-Hy$	$-y$
bc	*b'c'*	I_{bc}	$\frac{Px}{2} - H(h_{ab} + x \tan\theta)$	$-(h_{ab} + x \tan\theta)$
co	*co'*	$I_{cc'}$	$\frac{Px}{2} - Hh$	$-h$

Substituting the proper values in the general equation of condition

$$\frac{2}{EI_{ab}}\int_{o}^{h_{ab}} Hy^2 dy - \frac{2\sec\theta}{EI_{bc}}\int_{o}^{l_{bc}}\left[\frac{Px}{2} - H(h_{ab} + x\tan\theta)\right](h_{ab} + x\tan\theta)\,dx$$

$$-\frac{2h}{EI_{cc'}}\int_{l_{bc}}^{l_{bc}+\frac{l_{cc'}}{2}}\left(\frac{Px}{2} - Hh\right)dx = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

Integrating equation (1)

$$\frac{2}{EI_{ab}}\left[\frac{Hy^3}{3}\right]_{o}^{h} - \frac{2\sec\theta}{EI_{bc}}\left[\frac{Px^2 h_{ab}}{4} + \frac{Px^3}{6}\tan\theta - H\left(h_{ab}^2 x + h_{ab}x^2\tan\theta + \frac{x^3}{3}\tan^2\theta\right)\right]_{o}^{l_{bc}}$$

$$-\frac{2h}{EI_{cc'}}\left[\frac{Px^2}{4} - Hhx\right]_{l_{bc}}^{l_{bc}+\frac{l_{cc'}}{2}} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (2)$$

Substituting the limits in equation (2)

$$\frac{2Hh_{ab}^3}{3I_{ab}} - \frac{2\sec\theta}{I_{bc}}\left[\frac{Pl_{bc}^2 h_{ab}}{4} + \frac{Pl_{bc}^3}{6}\tan\theta - H\left(h_{ab}^2 l_{bc} + h_{ab}l_{bc}^2\tan\theta + \frac{l_{bc}^3}{3}\tan^2\theta\right)\right]$$

$$-\frac{2h}{I_{cc'}}\left[\frac{P}{4}\left(l_{bc}l_{cc'} + \frac{l_{cc'}^2}{4}\right) - Hh\frac{l_{cc'}}{2}\right] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Collecting the terms involving H and those involving P

$$\frac{2h^3_{ab}H}{3I_{ab}}+\frac{2sec\,\theta}{I_{bc}}\left(h^2_{ab}l_{bc}+h_{ab}l^2_{bc}tan\,\theta+\frac{l^3_{bc}}{3}\,tan^2\theta\right)H+\frac{2h^2l_{cc}\,H}{2I_{cc}}$$

$$=\frac{2sec\,\theta\;P}{I_{bc}}\left[\frac{l^2_{bc}h_{ab}}{4}+\frac{l^3_{bc}}{6}tan\,\theta\right]+\frac{2h}{I_{cc'}}\frac{P}{4}\left(l_{bc}\;l_{cc'}+\frac{l^2_{cc'}}{4}\right)\quad .\quad .\quad .\quad (4)$$

Solving for H,

$$H=\frac{\dfrac{2sec\,\theta}{I_{bc}}\left[\dfrac{l^2_{bc}h_{ab}}{4}+\dfrac{l^3_{bc}}{6}tan\,\theta\right]+\dfrac{2h}{4I_{cc'}}\left(l_{bc}\;l_{cc'}+\dfrac{l^2_{cc'}}{4}\right)}{\dfrac{2h^3_{ab}}{3I_{ab}}+\dfrac{2sec\,\theta}{I_{bc}}\left(h^2_{ab}l_{bc}+h_{ab}l^2_{bc}tan\,\theta+\dfrac{l^3_{bc}}{3}\,tan^2\theta\right)+\dfrac{h^2l_{cc'}}{I_{cc'}}}P\quad .\quad .\quad .\quad (5)$$

Substituting $l_{bc}\;sec\,\theta=S_{bc}$ and $h_{bc}=l_{bc}\;tan\,\theta$

$$H=\frac{\dfrac{hl_{cc'}}{8I_{cc'}}\left(4l_{bc}+l_{cc'}\right)+\dfrac{S_{bc}\;l_{bc}}{I_{bc}}\left[\dfrac{h_{ab}}{2}+\dfrac{h_{bc}}{3}\right]}{\dfrac{2h^3_{ab}}{3\,I_{ab}}+\dfrac{h^2l_{cc'}}{I_{cc'}}+\dfrac{2S_{bc}}{I_{bc}}\left(h^2_{ab}+h_{ab}h_{bc}+\dfrac{h^2_{bc}}{3}\right)}P\quad .\quad .\quad .\quad .\quad .\quad .\quad (6)$$

Fig. 7 gives sketches of the inverted U-frame having the lower ends of the columns fixed. Equations of the statically indeterminates, the horizontal reaction, and the bending moment at the lower end of the column, as determined from analyses, are given in the figure.

Knowing these indeterminates, the moments, and the forces at any section of the frame may be determined by ordinary analysis. Thus the moment at the middle of the top beam for the frame in the upper left-hand corner of Fig. 7 is equal to the algebraic sum of the bending moment at the lower end of the column, M_a, the moment of the horizontal reaction H with a moment arm equal to the height of the frame, the moment of the vertical reaction V about the section considered, and the moment of one load $\frac{P}{2}$ about the section. Fig. 7 indicates the manner in which the moment varies along the members composing the frame.

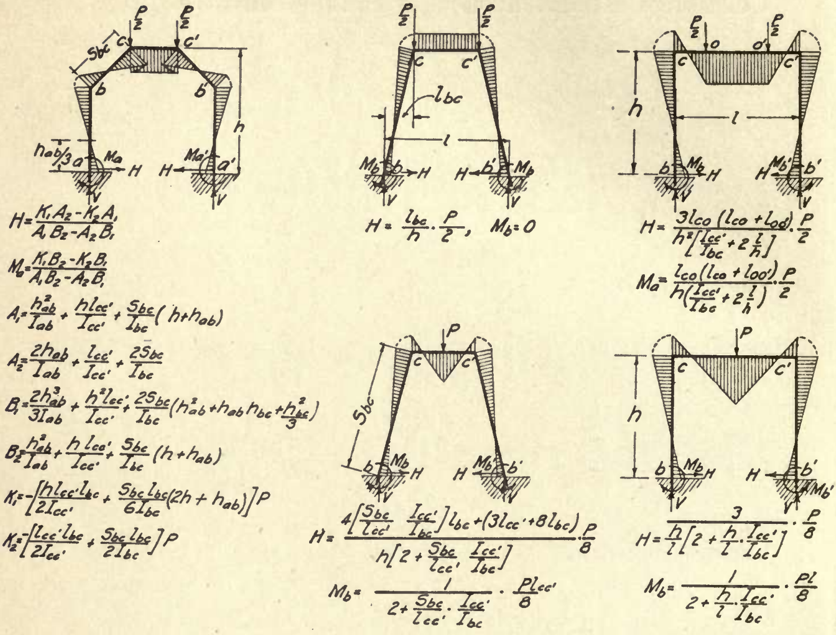

$\frac{P}{2}$
$\frac{P}{2}$
S_{bc}
c
c'
b
b'
h
$h_{ab}/3$
a
a'
M_a
M_a'
H
V
$H=\frac{K_1A_2-K_2A_1}{A_1B_2-A_2B_1}$
$M_a=\frac{K_1B_2-K_2B_1}{A_1B_2-A_2B_1}$
$A_1=\frac{h_{ab}^2}{I_{ab}}+\frac{hl_{cc'}}{I_{cc'}}+\frac{S_{bc}}{I_{bc}}(h+h_{ab})$
$A_2=\frac{2h_{ab}}{I_{ab}}+\frac{l_{cc'}}{I_{cc'}}+\frac{2S_{bc}}{I_{bc}}$
$B_1=\frac{2h_{ab}^3}{3I_{ab}}+\frac{h^2l_{cc'}}{I_{cc'}}+\frac{2S_{bc}}{I_{bc}}(h_{ab}^2+h_{ab}h_{bc}+\frac{h_{bc}^2}{3})$
$B_2=\frac{h_{ab}^2}{I_{ab}}+\frac{h\,l_{cc'}}{I_{cc'}}+\frac{S_{bc}}{I_{bc}}(h+h_{ab})$
$K_1=-\left[\frac{hl_{cc'}l_{bc}}{2I_{cc'}}+\frac{S_{bc}l_{bc}}{6I_{bc}}(2h+h_{ab})\right]P$
$K_2=-\left[\frac{l_{cc'}l_{bc}}{2I_{cc'}}+\frac{S_{bc}l_{bc}}{2I_{bc}}\right]P$
l_{bc}
l
M_b
b
b'
$H=\frac{l_{bc}}{h}\cdot\frac{P}{2},\quad M_b=0$
o
o'
$H=\frac{3l_{co}(l_{co}+l_{oo'})}{h^2\left[\frac{I_{cc'}}{I_{bc}}+2\frac{l}{h}\right]}\cdot\frac{P}{2}$
$M_a=\frac{l_{co}(l_{co}+l_{oo'})}{h\left(\frac{I_{cc'}}{I_{bc}}+2\frac{l}{h}\right)}\cdot\frac{P}{2}$
P
S_{bc}
M_b'
$H=\frac{4\left[\frac{S_{bc}}{l_{cc'}}\ \frac{I_{cc'}}{I_{bc}}\right]l_{bc}+(3l_{cc'}+8l_{bc})}{h\left[2+\frac{S_{bc}}{l_{cc'}}\ \frac{I_{cc'}}{I_{bc}}\right]}\cdot\frac{P}{8}$
$M_b=\frac{1}{2+\frac{S_{bc}}{l_{cc'}}\cdot\frac{I_{cc'}}{I_{bc}}}\cdot\frac{Pl_{cc'}}{8}$
$H=\frac{3}{\frac{h}{l}\left[2+\frac{h}{l}\cdot\frac{I_{cc'}}{I_{bc}}\right]}\cdot\frac{P}{8}$
$M_b=\frac{1}{2+\frac{h}{l}\cdot\frac{I_{cc'}}{I_{bc}}}\cdot\frac{Pl}{8}$

FIG. 7. SIMPLE FRAMES UNDER VERTICAL LOAD; LOWER ENDS OF COLUMNS FIXED

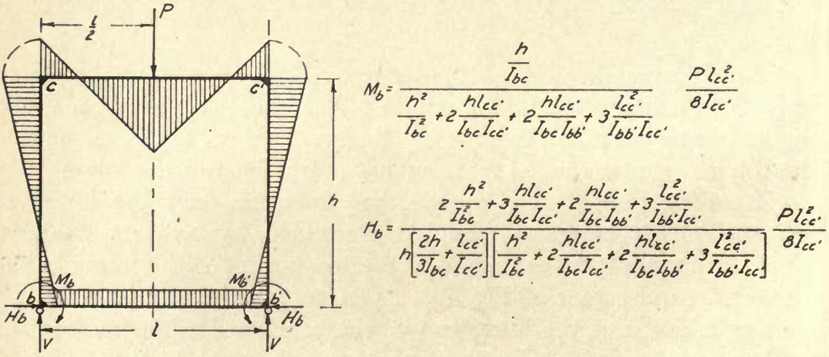

$\frac{l}{2}$
P
c
c'
h
M_b
M_b'
b
b'
H_b
l
V
$M_b=\frac{\frac{h}{I_{bc}}}{\frac{h^2}{I_{bc}^2}+2\frac{hl_{cc'}}{I_{bc}I_{cc'}}+2\frac{hl_{cc'}}{I_{bc}I_{bb'}}+3\frac{l_{cc'}^2}{I_{bb'}I_{cc'}}}\ \frac{Pl_{cc'}^2}{8I_{cc'}}$
$H_b=\frac{2\frac{h^2}{I_{bc}^2}+3\frac{hl_{cc'}}{I_{bc}I_{cc'}}+2\frac{hl_{cc'}}{I_{bc}I_{bb'}}+3\frac{l_{cc'}^2}{I_{bb'}I_{cc'}}}{h\left[\frac{2h}{3I_{bc}}+\frac{l_{cc'}}{I_{cc'}}\right]\left[\frac{h^2}{I_{bc}^2}+2\frac{hl_{cc'}}{I_{bc}I_{cc'}}+2\frac{hl_{cc'}}{I_{bc}I_{bb'}}+3\frac{l_{cc'}^2}{I_{bb'}I_{cc'}}\right]}\ \frac{Pl_{cc'}^2}{8I_{cc'}}$

FIG. 8. RECTANGULAR FRAME WITH RIGIDLY CONNECTED TIE AT BASE OF COLUMNS

Fig. 8 shows a rectangular frame in which all the joints are rigid, the horizontal cross tie at the bottom of the frame having rigid connection to the columns. The equations of the indeterminates M_b and H_b

for this frame are given in the figure. The figure indicates the manner in which the moment varies along the members.

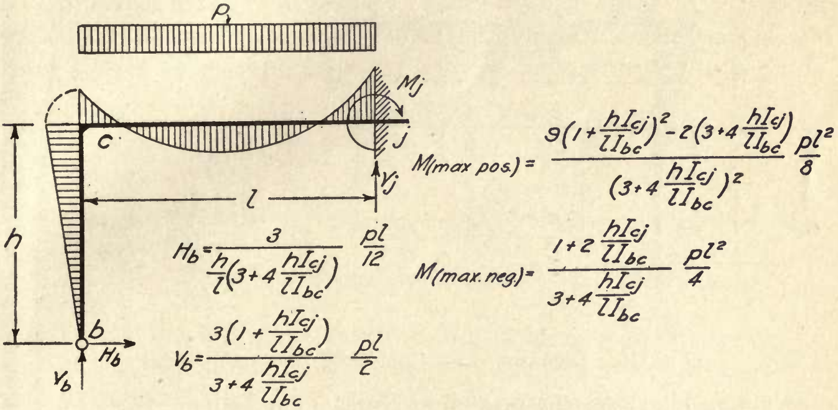

p
M_j
c
J
V_j
l
h
b
H_b
V_b
$H_b = \frac{3}{\frac{h}{l}\left(3+4\frac{hI_{cj}}{lI_{bc}}\right)}\frac{pl}{12}$
$V_b = \frac{3\left(1+\frac{hI_{cj}}{lI_{bc}}\right)}{3+4\frac{hI_{cj}}{lI_{bc}}}\frac{pl}{2}$
$M(max\ pos.) = \frac{9\left(1+\frac{hI_{cj}}{lI_{bc}}\right)^2-2\left(3+4\frac{hI_{cj}}{lI_{bc}}\right)}{\left(3+4\frac{hI_{cj}}{lI_{bc}}\right)^2}\frac{pl^2}{8}$
$M(max.\ neg.) = \frac{1+2\frac{hI_{cj}}{lI_{bc}}}{3+4\frac{hI_{cj}}{lI_{bc}}}\frac{pl^2}{4}$

FIG. 9. L-FRAME WITH COLUMN HINGED AT SUPPORT

Fig. 9 shows an unsymmetrical frame, here termed an L-frame, under uniform load and with the column hinged at the support. For this frame the horizontal and vertical reactions H_b and V_b at the column end are given in the figure.

The maximum positive bending moment in this frame occurs at the distance x from c where

$$x = \frac{3\left(1+\frac{hI_{cj}}{lI_{bc}}\right)}{3+\frac{4hI_{cj}}{lI_{bc}}}\ \frac{l}{2}$$

The value of the maximum positive bending moment is given in the figure. The maximum bending moment is the negative moment at the wall, which is also given in the figure.

Fig. 10 shows an L-frame in which the column is fixed at the support. On account of the fixity of the column there are three statically indeterminates for this frame, H_b, V_b, and M_b or M_j. The values of these are given in the figure.

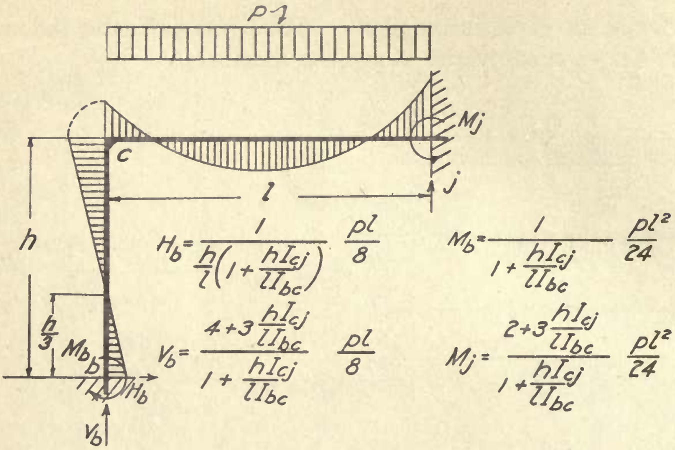

pl
Mj
c
j
l
h
$\frac{h}{3}$
M_b
H_b
V_b
$H_b = \frac{1}{\frac{h}{l}\left(1+\frac{hI_{cj}}{lI_{bc}}\right)} \frac{pl}{8}$
$M_b = \frac{1}{1+\frac{hI_{cj}}{lI_{bc}}} \frac{pl^2}{24}$
$V_b = \frac{4+3\frac{hI_{cj}}{lI_{bc}}}{1+\frac{hI_{cj}}{lI_{bc}}} \frac{pl}{8}$
$M_j = \frac{2+3\frac{hI_{cj}}{lI_{bc}}}{1+\frac{hI_{cj}}{lI_{bc}}} \frac{pl^2}{24}$

FIG. 10. L-FRAME WITH COLUMN FIXED AT SUPPORT

The formulas for the maximum positive moment in the beam and the distance of the section of maximum positive moment from c are

$$M_{max\ pos} = \frac{3\left(1+\frac{3hI_{ci}}{4lI_{bc}}\right)^2 - 2\left(1+\frac{hI_{ci}}{lI_{bc}}\right)}{\left(1+\frac{hI_{ci}}{lI_{bc}}\right)^2} \frac{pl^2}{24}$$

$$x = \frac{1+\frac{3hI_{ci}}{4lI_{bc}}}{1+\frac{hI_{ci}}{lI_{bc}}} \frac{l}{2}$$

The manner in which the bending moment varies along the members composing the frames is indicated in Fig. 9 and 10.

9. *Single Story Construction with Three Spans.*—In the design of a beam-and-girder or a flat-slab construction of a single story, many engineers do not take the effect of the bending of columns on the moments in other portions of the structure into consideration. Authors also have tried to analyze the stress distribution without taking this bending into account. Obviously a bending in the columns will allow an increased bending moment at the center of the span of a girder or slab loaded unsymmetrically with respect to the column, and stresses in the slab will be modified by variations in the ratio of the moment of inertia of the girder or slab to that of the column and of the column

height to the span length. Tests have shown that columns may be subjected to severe bending, and it will be seen that this will be more important for a single story structure than for others.

In actual cases, there may be twenty or more spans in succession, with different span lengths and different cross-sections of members, and consequently an exact analysis is hardly possible with any assumption.

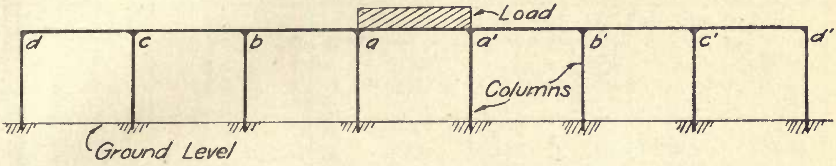


Fig. 11. Continuous Single Story Spans; One Panel Loaded

If panel *aa′* in Fig. 11 is loaded, the bending moments in slab *bc*, *cd*, *b′c′*, and *c′d′* are so small as to be negligible in actual cases. It may be assumed, therefore, that the end condition of slabs or beams *ab* and *a′b′* will, perhaps, be between the hinged and the fixed state at *b* and *b′*, the degree of fixity depending upon the ratio of moments of inertia of the column and the slab at that joint. Formulas will, therefore, be given for both conditions.

Fig. 12 shows the manner in which the bending moment varies along the members composing the frame for four combinations of end

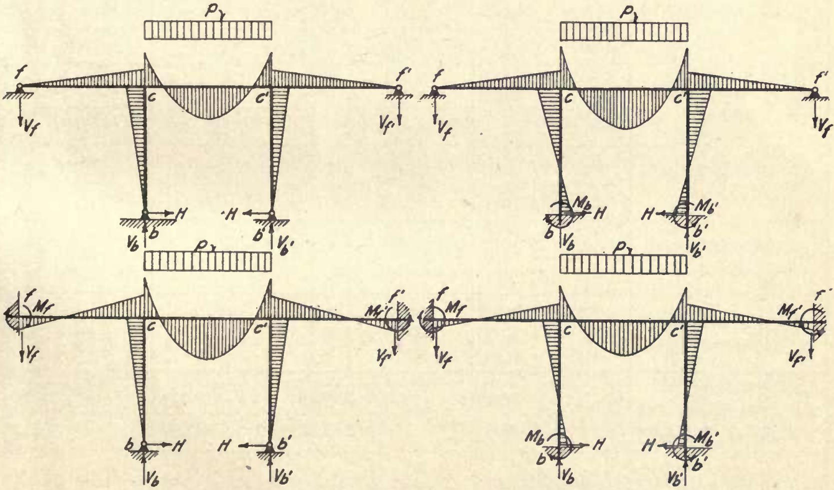


Fig. 12. Single Story Three-Span Frame, Having Middle Span Uniformly Loaded

conditions of beams and columns outside the loaded panel. The formulas which have been derived for these four cases are given in Table 3.

TABLE 3

FORMULAS FOR REACTIONS, BENDING MOMENTS, AND POINTS OF INFLECTION FOR SINGLE STORY THREE-SPAN FRAMES

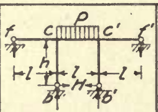
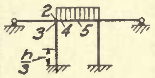
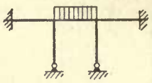
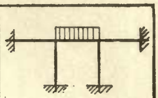
				
H Hor. reaction at col. ends	$\frac{1}{4n(3+5mn)}pl$	$\frac{1}{2n(4+5mn)}pl$	$\frac{1}{12n(1+2mn)}pl$	$\frac{1}{4n(2+3mn)}pl$
V_f Vert. reaction at beam ends	$\frac{mn}{4(3+5mn)}pl$	$\frac{mn}{4(4+5mn)}pl$	$\frac{mn}{6(1+2mn)}pl$	$\frac{mn}{4(2+3mn)}pl$
V_b Vert. reaction at col. ends	$\frac{6+11mn}{4(3+5mn)}pl$	$\frac{8+11mn}{4(4+5mn)}pl$	$\frac{3+7mn}{6(1+2mn)}pl$	$\frac{4+7mn}{4(2+3mn)}pl$
M_f Moment at beam ends	0	0	$\frac{mn}{(1+2mn)}\frac{pl^2}{18}$	$\frac{mn}{(2+3mn)}\frac{pl^2}{12}$
M_b Moment at column ends	0	$\frac{1}{(4+5mn)}\frac{pl^2}{6}$	0	$\frac{1}{(2+3mn)}\frac{pl^2}{12}$
M_2 Moment at point 2	$\frac{-mn}{(3+5mn)}\frac{pl^2}{4}$	$\frac{-mn}{(4+5mn)}\frac{pl^2}{4}$	$\frac{-mn}{(1+2mn)}\frac{pl^2}{9}$	$\frac{-mn}{(2+3mn)}\frac{pl^2}{6}$
M_3 Moment at point 3	$\frac{-1}{(3+5mn)}\frac{pl^2}{4}$	$\frac{-1}{(4+5mn)}\frac{pl^2}{3}$	$\frac{-1}{(1+2mn)}\frac{pl^2}{12}$	$\frac{-1}{(2+3mn)}\frac{pl^2}{6}$
M_4 Moment at point 4	$\frac{-(1+mn)}{(3+5mn)}\frac{pl^2}{4}$	$\frac{-(4+3mn)}{(4+5mn)}\frac{pl^2}{12}$	$\frac{-(3+4mn)}{3(1+2mn)}\frac{pl^2}{12}$	$\frac{-(1+mn)}{(2+3mn)}\frac{pl^2}{6}$
M_5 Moment at point 5	$\frac{(1+3mn)}{(3+5mn)}\frac{pl^2}{8}$	$\frac{(4+9mn)}{(4+5mn)}\frac{pl^2}{24}$	$\frac{(3+10mn)}{3(1+2mn)}\frac{pl^2}{24}$	$\frac{(2+5mn)}{(2+3mn)}\frac{pl^2}{24}$
Height of point of inflection in col.	0	$\frac{h}{3}$	0	$\frac{h}{3}$
Distance from column to point of inflect. in cent. span	$\frac{l}{2}\left[1\pm\sqrt{1-\frac{2(1+mn)}{(3+5mn)}}\right]$	$\frac{l}{2}\left[1\pm\sqrt{1-\frac{2(4+3mn)}{3(4+5mn)}}\right]$	$\frac{l}{2}\left[1\pm\sqrt{1-\frac{2(3+4mn)}{9(1+2mn)}}\right]$	$\frac{l}{2}\left[1\pm\sqrt{1-\frac{4(1+mn)}{3(2+3mn)}}\right]$
$m=\frac{I_{cc'}}{I_{bc}}$ $n=\frac{h}{l}$				

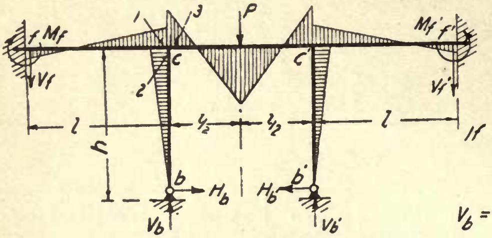
TABLE 4

COEFFICIENTS OF BENDING MOMENT FOR SINGLE STORY THREE-SPAN FRAME

α = coefficient of pl^2 for bending moment at end of middle span.
β = coefficient of pl^2 for bending moment at top of column.

	Coefficient	Values of m $m=\frac{I_{cc'}}{I_{bc}}$	Values of n $n=\frac{h}{l}$					
			0.20	0.50	0.75	1.00	1.25	1.50
Extreme Ends of Columns and Beams Hinged	$\alpha=\frac{1+mn}{(3+5mn)4}$	0.5	.0785	.0735	.0705	.0682	.0664	.0648
		1.0	.0750	.0682	.0648	.0625	.0608	.0595
		1.5	.0722	.0648	.0615	.0595	.0582	.0570
		2.0	.0700	.0625	.0595	.0577	.0564	.0555
		2.5	.0682	.0608	.0582	.0564	.0554	.0546
		3.0	.0667	.0595	.0570	.0556	.0546	.0540
	$\beta=\frac{1}{(3+5mn)4}$	0.5	.0714	.0588	.0513	.0455	.0408	.0370
		1.0	.0625	.0455	.0370	.0312	.0270	.0238
		1.5	.0556	.0370	.0290	.0238	.0202	.0175
		2.0	.0500	.0312	.0238	.0192	.0161	.0139
		2.5	.0455	.0270	.0202	.0161	.0134	.0115
		3.0	.0417	.0238	.0175	.0139	.0115	.0098
Ext. Ends of Beams Hinged Lower Ends of Columns Fixed	$\alpha=\frac{4+3mn}{(4+5mn)12}$	0.5	.0796	.0754	.0728	.0706	.0687	.0672
		1.0	.0767	.0706	.0672	.0647	.0630	.0616
		1.5	.0743	.0672	.0637	.0616	.0600	.0587
		2.0	.0723	.0648	.0616	.0595	.0581	.0572
		2.5	.0706	.0630	.0600	.0581	.0568	.0559
		3.0	.0690	.0616	.0588	.0571	.0559	.0550
	$\beta=\frac{1}{(4+5mn)3}$	0.5	.0741	.0635	.0575	.0513	.0468	.0430
		1.0	.0667	.0513	.0430	.0371	.0326	.0290
		1.5	.0607	.0430	.0346	.0290	.0250	.0219
		2.0	.0555	.0371	.0290	.0238	.0202	.0175
		2.5	.0513	.0326	.0250	.0202	.0170	.0147
		3.0	.0477	.0290	.0219	.0175	.0147	.0126
Extreme Ends of Columns and Beams Fixed	$\alpha=\frac{1+mn}{(2+3mn)6}$	0.5	.0798	.0757	.0732	.0714	.0700	.0686
		1.0	.0774	.0714	.0686	.0667	.0652	.0640
		1.5	.0745	.0686	.0658	.0640	.0628	.0619
		2.0	.0728	.0667	.0640	.0624	.0614	.0606
		2.5	.0714	.0652	.0626	.0614	.0604	.0598
		3.0	.0702	.0640	.0619	.0606	.0598	.0591
	$\beta=\frac{1}{(2+3mn)6}$	0.5	.0695	.0556	.0476	.0416	.0370	.0333
		1.0	.0595	.0416	.0333	.0278	.0238	.0208
		1.5	.0521	.0333	.0256	.0208	.0175	.0151
		2.0	.0463	.0278	.0208	.0167	.0139	.0119
		2.5	.0417	.0238	.0175	.0139	.0115	.0098
		3.0	.0379	.0208	.0151	.0119	.0098	.0077

In Table 4 are given values of the bending moment coefficients for both hinged and fixed ends at top of column and at end of middle span for six ratios of moments of inertia.



$$M_3 = -\frac{3+4\dfrac{h_{fc}}{lI_{bc}}}{3+2\dfrac{hI_{cc'}}{lI_{bc}}+4\dfrac{hI_{fc}}{lI_{bc}}}\;\frac{Pl}{8}$$

$$V_b = \frac{12+8\dfrac{hI_{cc'}}{lI_{bc}}+22\dfrac{hI_{fc}}{lI_{bc}}}{3+2\dfrac{hI_{cc'}}{lI_{bc}}+4\dfrac{hI_{fc}}{lI_{bc}}}\;\frac{P}{8}$$

$$V_f = \frac{3\dfrac{hI_{fc}}{lI_{bc}}}{3+2\dfrac{hI_{cc'}}{lI_{bc}}+4\dfrac{hI_{fc}}{lI_{bc}}}\;\frac{P}{4}$$

$$H_b = \frac{3}{\dfrac{h}{l}\left(3+2\dfrac{hI_{cc'}}{lI_{bc}}+4\dfrac{hI_{fc}}{lI_{bo}}\right)}\;\frac{P}{8}$$

$$M_f = \frac{V_f l}{3}, \quad M_1 = \frac{-2V_f l}{3}, \quad M_2 = -H_b h, \quad M_3 = M_1 + M_2$$

If $I_{fc} = I_{cc'}$ these formulas become:

$$V_b = \frac{2+5\dfrac{hI_{cc'}}{lI_{bc}}}{1+2\dfrac{hI_{cc'}}{lI_{bc}}}\;\frac{P}{4}$$

$$V_f = \frac{\dfrac{hI_{cc'}}{lI_{bc}}}{1+2\dfrac{hI_{cc'}}{lI_{bc}}}\;\frac{P}{4}$$

$$H_b = \frac{1}{\left(1+2\dfrac{hI_{cc'}}{lI_{bc}}\right)\dfrac{h}{l}}\;\frac{P}{8}$$

$$M_3 = \frac{3+4\dfrac{hI_{cc'}}{lI_{bc}}}{3\left(1+2\dfrac{hI_{cc'}}{lI_{bc}}\right)}\;\frac{Pl}{8}$$

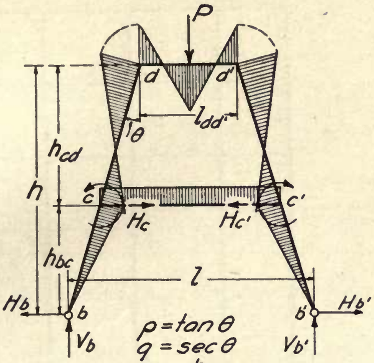
FIG. 13. SINGLE STORY THREE-SPAN FRAME UNDER CONCENTRATED LOAD

For a three-span frame with concentrated load at the center of the middle panel, Fig. 13 shows the manner in which the moment varies along the several members composing the frame.

Assuming symmetry about the vertical center line of this frame the formulas for the horizontal and vertical reactions are given in Fig. 13, and also formulas for bending moments at top of column and at end of middle span.

10. *Trestle Bent with Tie.*—The frame shown in Fig. 14, frequently termed the A-frame, may be used in trestle construction. Formulas for $M_{cc'}$, H_b, and H_c are given in the figure.

Knowing the values of $M_{cc'}$, H_c, and H_b, the stresses at any section of the frame may be computed. When $\theta = 0$, $I_{bc} = 0$, and l_{bc} approaches 0, making $h_{cd} = h$ (that is when $k = 1$), this frame becomes the same as that shown in Fig. 8 and the formulas for $M_{cc'}$ and H_c reduce to the same form as those given in Fig. 8.



$$p = \tan\theta$$
$$q = \sec\theta$$
$$K = \frac{h_{cd}}{h}$$

$$M_{cc'} = \frac{\dfrac{h\,l_{dd'}}{I_{cd}I_{dd'}}}{\dfrac{h l_{dd'} l_{cc'}}{I_{cd}I_{dd'}l_{cc'}}\left(\dfrac{3I_{bc}}{qh(1-K)}+\dfrac{2I_{cc'}}{l_{cc'}}+\dfrac{3I_{cd}}{qKh}\right)+\dfrac{qKh^2 l_{cc'}}{2I_{cd}^2 I_{cc'}}\left(\dfrac{3I_{bc}}{qh(1-K)}+\dfrac{2I_{cc'}}{l_{cc'}}+\dfrac{4I_{cd}}{qhK}\right)}\;\frac{Pl_{dd'}}{8}$$

$$H_c = \frac{\dfrac{l_{dd'}l_{cc'}}{I_{dd'}I_{cc'}}\left[\dfrac{I_{bc}}{I_{cd}}\dfrac{(3-2K)}{2q(1-K)^2}+\dfrac{1}{qK}+\dfrac{hI_{cc'}}{I_{cd}l_{cc'}}\right]}{\dfrac{h l_{dd'} l_{cc'}}{I_{cd}I_{dd'}I_{cc'}}\left(\dfrac{3I_{bc}}{qh(1-K)}+\dfrac{2I_{cc'}}{l_{cc'}}+\dfrac{3I_{cd}}{qKh}\right)+\dfrac{qKh^2 l_{cc'}}{2I_{cd}^2 I_{cc'}}\left(\dfrac{3I_{bc}}{qh(1-K)}+\dfrac{2I_{cc'}}{l_{cc'}}+\dfrac{4I_{cd}}{qKh}\right)}\;\frac{3Pl_{dd'}}{8Kh}$$

$$H_b = \frac{\dfrac{l_{dd'}}{I_{dd'}}\left[\dfrac{l_{cc'}}{I_{cc'}}\left(\dfrac{3l_{dd'}}{8qh(1-K)^2}\dfrac{I_{bc}}{I_{cd}}-\dfrac{3p}{q(1-K)}\dfrac{I_{bc}}{I_{cd}}-\dfrac{3p}{qK}\right)-\dfrac{2ph}{I_{cd}}\right]-\dfrac{ph}{I_{cd}}\left[\dfrac{l_{cc'}}{I_{cc'}}\left(\dfrac{3KI_{bc}}{2(1-K)I_{cd}}+2\right)+\dfrac{qhK}{I_{cd}}\right]}{\dfrac{h l_{dd'} l_{cc'}}{I_{cd}I_{dd'}I_{cc'}}\left(\dfrac{3I_{bc}}{qh(1-K)}+\dfrac{2I_{cc'}}{l_{cc'}}+\dfrac{3I_{cd}}{qKh}\right)+\dfrac{qh^2K l_{cc'}}{2I_{cd}^2 I_{cc'}}\left(\dfrac{3I_{bc}}{qh(1-K)}+\dfrac{2I_{cc'}}{l_{cc'}}+\dfrac{4I_{cd}}{qKh}\right)}\;\frac{P}{2}$$

FIG. 14. A-FRAME SHOWING FORM OF MOMENT CURVE

11. *Building Construction with Several Stories and Spans.*—In the actual construction of buildings of reinforced concrete, it is common to use a continuous slab for floors supported by a number of columns. A loading which produces serious bending in columns is, of course, an eccentric arrangement, such as is shown in Fig. 15. The moments of inertia of columns are sometimes smaller than those of slabs. Accordingly, the bending moment in the floor slab $fcc'f'$ is greatly modified by the flexure of columns bc, $b'c'$, cd, and $c'd'$.

In present practice, frequently little attention is paid to this point, and columns are assumed to be rigid enough to resist the bending. This assumption may be approximately true for the lower stories,

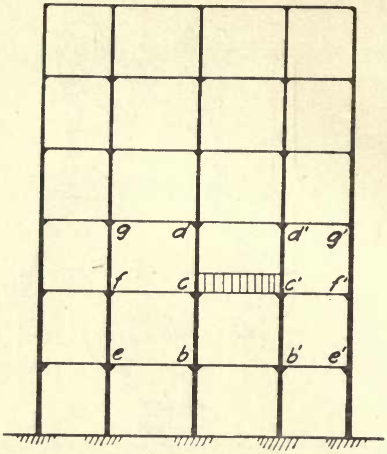


FIG. 15. CROSS-SECTION OF BUILDING FRAME WITH SINGLE PANEL LOADED

where the columns have large diameters, but it is not true for the upper stories, where the cross-section of columns is usually small, and serious bending stress may exist in the column due to eccentric loading. An exact analysis is hardly possible because there are many unknown conditions entering into the solution. From a practical standpoint, it is easily understood that the bending moment in floor slabs, *gdd'g'* and *ebb'e'*, (see Fig. 15) due to the load on the floor *cc'* is so small as to be inconsiderable if the floors are of moderate thickness. That is, the columns *cb*, *c'b'*, *cd*, and *c'd'* are practically fixed at *b*, *b'*, *d*, and *d'*, respectively. If the floor slabs are not thick enough to keep the

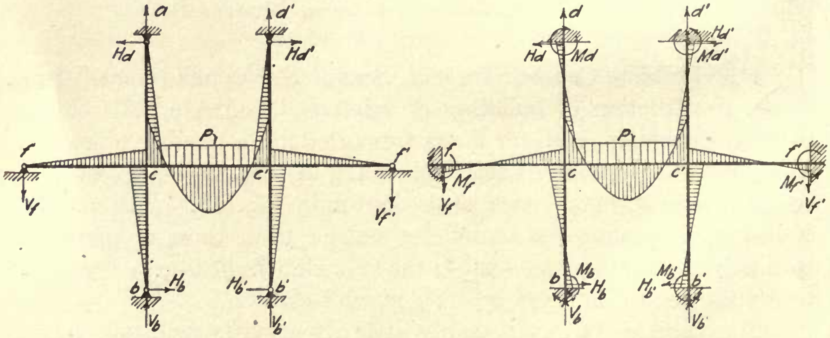


FIG. 16. TWO-STORY THREE-SPAN FRAME HAVING MIDDLE SPAN UNIFORMLY LOADED

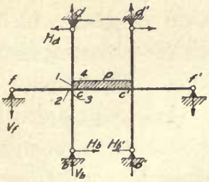
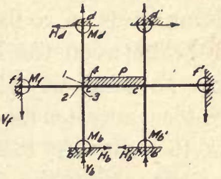


TABLE 5

FORMULAS FOR TWO-STORY THREE-SPAN FRAMES



ENDS HINGED

	GENERAL CASE (a)	SPECIAL CASE (b) $l_{cc'}=l_{fc}=l.\quad I_{cc'}=I_{fc}$	SPECIAL CASE (c) $h_{cd}=h_{bc}=l_{fc}=l_{cc'}=l$ $I_{cc'}=I_{bc}=I_{fc}=I_{cd}$
H_b = Horizontal reaction at b	$\dfrac{1}{\frac{h_{bc}}{l_{cc'}}\left(3+3\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+3\frac{h_{bc}I_{fc}}{l_{fc}I_{bc}}+2\frac{h_{bc}I_{cc'}}{l_{cc'}I_{bc}}\right)}\;\dfrac{pl_{cc'}}{4}$	$\dfrac{1}{\frac{h_{bc}}{l}\left(3+3\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+5\frac{h_{bc}I_{cc'}}{lI_{bc}}\right)}\;\dfrac{pl}{4}$	$\dfrac{pl}{44}$
H_d = Horizontal reaction at d	$\dfrac{1}{\frac{h_{cd}}{l_{cc'}}\left(3+3\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+3\frac{h_{cd}I_{fc}}{l_{fc}I_{cd}}+2\frac{h_{cd}I_{cc'}}{l_{cc'}I_{cd}}\right)}\;\dfrac{pl_{cc'}}{4}$	$\dfrac{1}{\frac{h_{cd}}{l}\left(3+3\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+5\frac{h_{cd}I_{cc'}}{lI_{cd}}\right)}\;\dfrac{pl}{4}$	$\dfrac{pl}{44}$
V_f = Vertical reaction at f	$\dfrac{1}{\frac{l_{fc}}{l_{cc'}}\left(3+3\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+3\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+2\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}\right)}\;\dfrac{pl_{cc'}}{4}$	$\dfrac{1}{5+3\frac{l\,I_{cd}}{h_{cd}I_{cc'}}+3\frac{l\,I_{bc}}{h_{bc}I_{cc'}}}\;\dfrac{pl}{4}$	$\dfrac{pl}{44}$
V_b = Vertical reaction at b	$\dfrac{6+\frac{l_{cc'}}{l_{fc}}+6\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+6\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+4\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}}{3+3\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+3\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+2\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}}\;\dfrac{pl_{cc'}}{8}$	$\dfrac{11+6\frac{l}{I_{cc'}}\left(\frac{I_{cd}}{h_{cd}}+\frac{I_{bc}}{h_{bc}}\right)}{5+3\frac{l}{I_{cc'}}\left(\frac{I_{cd}}{h_{cd}}+\frac{I_{bc}}{h_{bc}}\right)}\;\dfrac{pl}{8}$	$23\dfrac{pl}{88}$
M_1 = Bending moment at 1	$\dfrac{1}{3+3\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+3\frac{h_{cd}I_{fc}}{l_{fc}I_{cd}}+2\frac{h_{cd}I_{cc'}}{l_{cc'}I_{cd}}}\;\dfrac{pl^2_{cc'}}{4}$	$\dfrac{1}{3+3\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+5\frac{h_{cd}I_{cc'}}{lI_{cd}}}\;\dfrac{pl^2}{4}$	$\dfrac{pl^2}{44}$
M_2 = Bending moment at 2	$\dfrac{1}{3+3\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+3\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+2\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}}\;\dfrac{pl^2_{cc'}}{4}$	$\dfrac{1}{5+3\frac{l}{I_{cc'}}\left(\frac{I_{cd}}{h_{cd}}+\frac{I_{bc}}{h_{bc}}\right)}\;\dfrac{pl^2}{4}$	$\dfrac{pl^2}{44}$
M_3 = Bending moment at 3	$\dfrac{1}{3+3\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+3\frac{h_{bc}I_{fc}}{l_{fc}I_{bc}}+2\frac{h_{bc}I_{cc'}}{l_{cc'}I_{bc}}}\;\dfrac{pl^2_{cc'}}{4}$	$\dfrac{1}{3+3\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+5\frac{h_{bc}I_{cc'}}{lI_{bc}}}\;\dfrac{pl^2}{4}$	$\dfrac{pl^2}{44}$
M_4 = Bending moment at 4	$M_1+M_2+M_3$	$M_1+M_2+M_3$	$3\dfrac{pl^2}{44.}$

ENDS FIXED

	GENERAL CASE (a)	SPECIAL CASE (b) $l_{cc'}=l_{fc}=l.\quad I_{cc'}=I_{fc}$	SPECIAL CASE (c) $h_{cd}=h_{bc}=l_{fc}=l_{cc'}=l$ $I_{cc'}=I_{fc}=I_{bc}=I_{cd}$
M_b = Moment at col ends b & b'	$\dfrac{1}{2+2\frac{h_{bc}I_{fc}}{l_{fc}I_{bc}}+2\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+\frac{h_{bc}I_{cc'}}{l_{cc'}I_{bc}}}\;\dfrac{pl^2_{cc'}}{12}$	$\dfrac{1}{2+2\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+3\frac{h_{bc}I_{cc'}}{lI_{bc}}}\;\dfrac{pl^2}{12}$	$\dfrac{pl^2}{84}$
M_d = Moment at col ends d & d'	$\dfrac{1}{2+2\frac{h_{cd}I_{fc}}{l_{fc}I_{cd}}+2\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+\frac{h_{cd}I_{cc'}}{l_{cc'}I_{cd}}}\;\dfrac{pl^2_{cc'}}{12}$	$\dfrac{1}{2+2\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+3\frac{h_{cd}I_{cc'}}{lI_{cd}}}\;\dfrac{pl^2}{12}$	$\dfrac{pl^2}{84}$
M_f = Moment at beam ends f & f'	$\dfrac{1}{2+2\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+2\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}}\;\dfrac{pl^2_{cc'}}{12}$	$\dfrac{1}{2\frac{l}{I_{cc'}}\left(\frac{I_{bc}}{h_{bc}}+\frac{I_{cd}}{h_{cd}}\right)+3}\;\dfrac{pl^2}{12}$	$\dfrac{pl^2}{84}$
H_b = Horizontal reaction at b & b'	$\dfrac{1}{\frac{h_{bc}}{l_{cc'}}\left(2+2\frac{h_{bc}I_{fc}}{l_{fc}I_{bc}}+2\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+\frac{h_{bc}I_{cc'}}{l_{cc'}I_{bc}}\right)}\;\dfrac{pl^2_{cc'}}{4}$	$\dfrac{1}{\frac{h_{bc}}{l}\left(2+2\frac{h_{bc}I_{cd}}{h_{cd}I_{bc}}+3\frac{h_{bc}I_{cc'}}{lI_{bc}}\right)}\;\dfrac{pl}{4}$	$\dfrac{pl}{28}$
H_d = Horizontal reaction at d & d'	$\dfrac{1}{\frac{h_{cd}}{l_{cc'}}\left(2+2\frac{h_{cd}I_{fc}}{l_{fc}I_{cd}}+2\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+\frac{h_{cd}I_{cc'}}{l_{cc'}I_{cd}}\right)}\;\dfrac{pl^2_{cc'}}{4}$	$\dfrac{1}{\frac{h_{cd}}{l}\left(2+2\frac{h_{cd}I_{bc}}{h_{bc}I_{cd}}+3\frac{h_{cd}I_{cc'}}{lI_{cd}}\right)}\;\dfrac{pl}{4}$	$\dfrac{pl}{28}$
V_f = Vertical reaction at f & f'	$\dfrac{1}{\frac{l_{fc}}{l_{cc'}}\left(2+2\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+2\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}\right)}\;\dfrac{pl_{cc'}}{4}$	$\dfrac{1}{3+2\frac{l}{I_{cc'}}\left(\frac{I_{bc}}{h_{bc}}+\frac{I_{cd}}{h_{cd}}\right)}\;\dfrac{pl}{4}$	$\dfrac{pl}{28}$
V_b = Vertical reaction at b & b'	$\dfrac{4+\frac{l_{cc'}}{l_{fc}}+4\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+4\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+2\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}}{2+2\frac{l_{fc}I_{bc}}{h_{bc}I_{fc}}+2\frac{l_{fc}I_{cd}}{h_{cd}I_{fc}}+\frac{l_{fc}I_{cc'}}{l_{cc'}I_{fc}}}\;\dfrac{pl_{cc'}}{8}$	$\dfrac{7+4\frac{l}{I_{cc'}}\left(\frac{I_{bc}}{h_{bc}}+\frac{I_{cd}}{h_{cd}}\right)}{2\frac{l}{I_{cc'}}\left(\frac{I_{bc}}{h_{bc}}+\frac{I_{cd}}{h_{cd}}\right)+3}\;\dfrac{pl}{8}$	$15\dfrac{pl}{56}$
M_1 Moment at 1	$2M_d$	$2M_d$	$\dfrac{pl^2}{42}$
M_2 " " 2	$2M_f$	$2M_f$	$\dfrac{pl^2}{42}$
M_3 " " 3	$2M_b$	$2M_b$	$\dfrac{pl^2}{42}$
M_4 " " 4	$2(M_b+M_f+M_d)$	$2(M_b+M_f+M_d)$	$\frac{3}{42}pl^2$

column ends in a fixed condition, the end condition of the columns will be between the hinged and the fixed state. Using these assumptions an analysis which is almost exact is possible. The resulting formulas may be used in the design of buildings. For such a frame, Fig. 16 shows the manner in which the moment varies along the members composing the frame for two combinations of end conditions of beams and columns. The formulas for the horizontal and vertical reactions and the bending moments are given in Table 5 for ends of columns and beams hinged and for ends of columns and beams fixed, the end spans being equal in both cases. Formulas are also given for equal spans and equal moments of inertia of the beams and for equal spans, story heights, and moments of inertia of beams and columns. Fig. 17 gives numerical values of the coefficients of pl^2 for the case in which ends of beams and columns are fixed, α being the coefficient of the bending moment at the top of the lower column and β the coefficient at the foot of the upper column. In Table 6 are given values

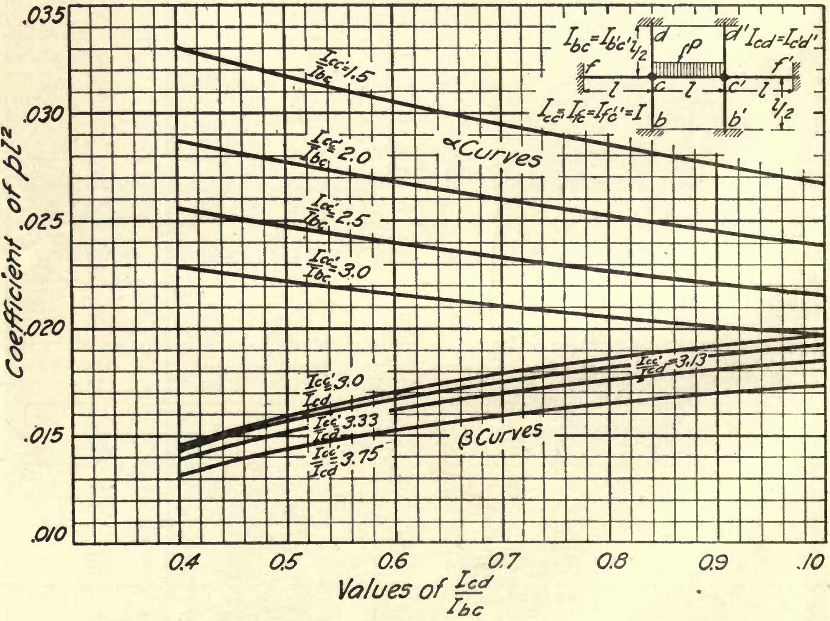


FIG. 17. COEFFICIENT OF BENDING MOMENTS FOR TOP OF LOWER COLUMN AND FOOT OF UPPER COLUMN FOR TWO-STORY THREE-SPAN FRAME HAVING ALL EXTERNAL CONNECTIONS FIXED

TABLE 6

COEFFICIENTS OF BENDING MOMENT FOR TWO-STORY THREE-SPAN FRAME WITH ENDS OF COLUMNS AND BEAMS FIXED

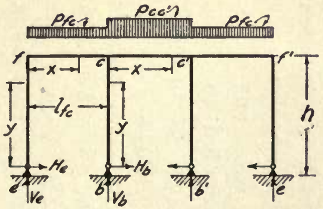
α = coefficient of pl^2 for bending moment at top of lower column.
β = coefficient of pl^2 for bending moment at foot of upper column.

	$\frac{I_{fc}}{I_{bc}}$	$\frac{I_{cd}}{I_{bc}}$			
		0.4	0.6	0.8	1.0
α	1.5	.0330	.0306	.0285	.0267
	2.0	.0287	.0269	.0253	.0238
	2.5	.0255	.0240	.0227	.0215
	3.0	.0228	.0216	.0206	.0196
β	$\frac{I_{fc}}{I_{cd}}$				
	3.75	.0132	.0153	.0165	.0173
	3.33	.0139	.0162	.0175	.0185
	3.13	.0143	.0167	.0181	.0192
	3.00	.0145	.0170	.0185	.0196

of the bending moment coefficients α and β for four ratios of the moments of inertia. The frame with hinged ends has nine statically indeterminate quantities, while the frame with fixed ends has fifteen statically indeterminates, but the condition of symmetrical loading shown greatly reduces the number of these quantities. In the analyses it has been assumed that the vertical reactions at b and d (also at b' and d') are the same. This assumption may not be the real condition in actual cases, but no effect is produced on bending moments by it.

12. *Frame with Three Spans.*—In bridge or trestle construction across a wide stream or valley, several spans may be built continuously as a monolith. Because of the necessity of providing expansion joints, the number of spans thus connected is frequently limited to three. Rigidly connected frames with three spans, equal or unequal, may advantageously be used for bridges of moderate spans.

No analytical formulas for such frames have, to the writer's knowledge, been published. Fig. 18 and 19 give the formulas for a three-span frame under various conditions of load. Fig. 20 shows the manner in which the moment varies along the members composing the frames. Table 7 gives values of the bending moment coefficients



Bending Moment Equations

Member	Bending Moment
ef	$M_{ef} = -H_e y$
fc	$M_{fc} = -H_e h - V_b x + \frac{2p_{fc}l_{fc} + p_{cc'}l_{cc'}}{2}x - \frac{p_{fc}x^2}{2}$
bc	$M_{bc} = -H_b y$
cc'	$M_{cc'} = -H_e h - H_b h - V_b l_{fc} + \frac{p_{fc}l_{fc}^2 + p_{cc'}(l_{cc'}l_{fc} + l_{cc'}x - x^2)}{2}$

General Case

$$H_b = \text{horizontal reaction at } b = \frac{p_{cc'}l_{cc'}\left(3\frac{l_{cc'}}{l_{fc}} + 4\frac{hl_{cc'}I_{fc}}{l_{fc}^2 I_{ef}}\right) - 3p_{fc}l_{fc}\left(1 + 2\frac{hI_{fc}}{l_{fc}I_{ef}}\right)}{4\frac{h}{l_{fc}}\Delta}$$

$$H_e = \text{horizontal reaction at } e = \frac{p_{fc}l_{fc}\left[3 + 6\frac{hI_{fc}}{l_{fc}I_{bc}} + 2\frac{hI_{cc'}}{l_{cc'}I_{bc}}\right] - p_{cc'}l_{cc'}\left(\frac{2hl_{cc'}I_{fc}}{l_{fc}^2 I_{bc}}\right)}{4\frac{h}{l_{fc}}\Delta}$$

$$\Delta = 9 + 12\frac{hI_{fc}}{l_{fc}I_{ef}} + 12\frac{hI_{fc}}{l_{fc}I_{bc}} + 6\frac{hI_{cc'}}{l_{cc'}I_{bc}} + 2\frac{hI_{fc}}{l_{fc}I_{ef}}\left(6\frac{hI_{fc}}{l_{fc}I_{bc}} + 4\frac{hI_{cc'}}{l_{cc'}I_{bc}}\right)$$

$$V_e = \left\{ \begin{aligned} &\frac{p_{fc}l_{fc}\left[9 + 9\frac{hI_{fc}}{l_{fc}I_{ef}} + 15\frac{hI_{fc}}{l_{fc}I_{bc}} + 6\frac{hI_{cc'}}{l_{cc'}I_{bc}} + 6\frac{h^2 I_{fc}}{l_{fc}I_{ef}}\left(2\frac{I_{fc}}{l_{fc}I_{bc}} + \frac{I_{cc'}}{l_{cc'}I_{bc}}\right)\right]}{2\Delta} \\ &- \frac{p_{cc'}l_{cc'}\left[\frac{hl_{cc'}I_{fc}}{l_{fc}^2 I_{bc}}\left(3 + 2\frac{hI_{fc}}{l_{fc}I_{ef}}\right)\right]}{2\Delta} \end{aligned} \right\}$$

$$V_b = \frac{p_{fc}l_{fc}\left[9 + 15\frac{hI_{fc}}{l_{fc}I_{ef}} + 9\frac{hI_{fc}}{l_{fc}I_{bc}} + 6\frac{hI_{cc'}}{l_{cc'}I_{bc}} + 2\frac{hI_{fc}}{l_{fc}I_{ef}}\left(6\frac{hI_{fc}}{l_{fc}I_{bc}} + 5\frac{hI_{cc'}}{l_{cc'}I_{bc}}\right)\right] + p_{cc'}l_{cc'}\left[9 + 12\frac{hI_{fc}}{l_{fc}I_{ef}} + 12\frac{hI_{fc}}{l_{fc}I_{bc}} + 6\frac{hI_{cc'}}{l_{cc'}I_{bc}} + 2\frac{hI_{fc}}{l_{fc}I_{ef}}\left(6\frac{hI_{fc}}{l_{fc}I_{bc}} + 4\frac{hI_{cc'}}{l_{cc'}I_{bc}}\right) + \frac{hl_{cc'}I_{fc}}{l_{fc}^2 I_{bc}}\left(3 + 2\frac{hI_{fc}}{l_{fc}I_{ef}}\right)\right]}{2\Delta}$$

Special Cases

If $l_{fc} = l_{cc'} = l \quad I_{fc} = I_{cc'} \quad I_{bc} = I_{ef} \quad p_{fc} = p_{cc'} = p$

$$H_b = -\frac{\frac{hI_{cc'}}{lI_{bc}}}{2\frac{h}{l}\Delta}pl \qquad H_e = \frac{3 + 6\frac{hI_{cc'}}{lI_{bc}}}{4\frac{h}{l}\Delta}pl$$

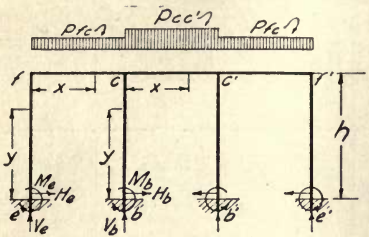
$$V_b = \frac{18 + 63\frac{hI_{cc'}}{lI_{bc}} + 44\left(\frac{hI_{cc'}}{lI_{bc}}\right)^2}{2\Delta}pl \qquad V_e = \frac{9 + 27\frac{hI_{cc'}}{lI_{bc}} + 16\left(\frac{hI_{cc'}}{lI_{bc}}\right)^2}{2\Delta}pl$$

If $I_{fc} = I_{cc'} = I_{bc} \quad l_{fc} = l_{cc'} = h \quad p_{fc} = p_{cc'} = p$:

$$H_b = -\frac{2}{236}pl \qquad H_e = \frac{9}{236}pl$$

$$V_b = \frac{250}{236}pl \qquad V_e = \frac{104}{236}pl$$

FIG. 18. FORMULAS FOR SINGLE STORY THREE-SPAN FRAME WITH FOUR COLUMNS HAVING COLUMNS HINGED AT LOWER END



General Case

Bending Moment Equations.

Member.	Bending Moment.
cc'	$M_{cc'} = -2M_e - 2M_b - V_b l_{fc} + l_{fc}\left(\frac{p_{fc} l_{fc} + p_{cc'} l_{cc'}}{2}\right) + \frac{p_{cc'} l_{cc'}}{2} x - \frac{p_{cc'} x^2}{2}$
fc	$M_{fc} = -2M_e + \left(p_{fc} l_{fc} + \frac{p_{cc'} l_{cc'}}{2}\right) x - V_b x - \frac{p_{fc} x^2}{2}$
ef	$M_{ef} = M_e\left(1 - \frac{3y}{h}\right)$
bc	$M_{bc} = M_b\left(1 - \frac{3y}{h}\right)$

$$M_e = \begin{matrix}\text{Bending}\\ \text{moment at } e\end{matrix} = \frac{p_{fc} l_{fc}^2\left[\frac{hI_{cc'}}{l_{cc'}I_{bc}} + \frac{3hI_{fc}}{l_{fc}I_{bc}} + 2\right] - p_{cc'} l_{cc'}^2\left(\frac{hI_{fc}}{l_{fc}I_{bc}}\right)}{12\Delta}$$

$$\Delta = 4 + 2\frac{hI_{cc'}}{l_{cc'}I_{bc}} + 4\frac{hI_{fc}}{l_{fc}I_{bc}} + 4\frac{hI_{fc}}{l_{fc}I_{ef}} + \frac{h^2 I_{fc} I_{cc'}}{l_{cc'} l_{fc} I_{ef} I_{bc}}\left(2 + 3\frac{l_{cc'} I_{fc}}{l_{fc} I_{cc'}}\right)$$

$$M_b = \begin{matrix}\text{Bending}\\ \text{moment at } b\end{matrix} = \frac{p_{cc'} l_{cc'}^2 2\left(\frac{hI_{fc}}{l_{fc}I_{ef}} + 1\right) - p_{fc} l_{fc}^2\left(2 + \frac{3hI_{fc}}{l_{fc}I_{ef}}\right)}{12\Delta}$$

$$M_e = \frac{H_e h}{3} \qquad M_b = \frac{H_b h}{3}$$

$$V_e = p_{fc} l_{fc} + \frac{p_{cc'} l_{cc'}}{2} - V_b$$

$$V_b = \frac{p_{fc} l_{fc}\left[8 + 10\frac{hI_{fc}}{l_{fc}I_{ef}} + 6\frac{hI_{fc}}{l_{fc}I_{bc}}\left(1 + \frac{hI_{fc}}{l_{fc}I_{ef}}\right) + \frac{hI_{cc'}}{l_{cc'}I_{bc}}\left(4 + 5\frac{hI_{fc}}{l_{fc}I_{ef}}\right)\right] + p_{cc'} l_{cc'}\left[8\left(1 + \frac{hI_{fc}}{l_{fc}I_{ef}}\right) + 4\frac{hI_{cc'}}{l_{cc'}I_{bc}}\left(1 + \frac{hI_{fc}}{l_{fc}I_{ef}}\right) + \frac{hI_{fc}}{l_{fc}I_{bc}}\left[\left(2 + \frac{hI_{fc}}{l_{fc}I_{ef}}\right)\left(6 + \frac{l_{cc'}}{l_{fc}}\right) - 4\right]\right]}{4\Delta}$$

Special Cases

If $l_{fc} = l_{cc'} = l$, $I_{fc} = I_{cc'}$, $I_{ef} = I_{bc}$, $p_{fc} = p_{cc'} = p$

$$M_e = \frac{2 + 3\frac{hI_{cc'}}{lI_{bc}}}{12\Delta} pl^2 \qquad M_b = -\frac{\frac{hI_{cc'}}{lI_{bc}}}{12\Delta} pl^2$$

$$V_b = \frac{16 + 42\frac{hI_{cc'}}{lI_{bc}} + 22\left(\frac{hI_{cc'}}{lI_{bc}}\right)^2}{4\Delta} pl \qquad \Delta = 4 + 10\frac{hI_{cc'}}{lI_{bc}} + 5\left(\frac{hI_{cc'}}{lI_{bc}}\right)^2$$

If $l_{fc} = l_{cc'} = l = h$, $I_{fc} = I_{cc'} = I_{bc}$, $p_{fc} = p_{cc'} = p$

$$M_e = \frac{5}{228} pl^2 \qquad M_b = -\frac{1}{228} pl^2$$

$$V_e = \frac{17}{38} pl \qquad V_b = \frac{20}{19} pl$$

FIG. 19. FORMULAS FOR SINGLE STORY THREE-SPAN FRAME WITH FOUR COLUMNS HAVING COLUMNS FIXED AT LOWER END

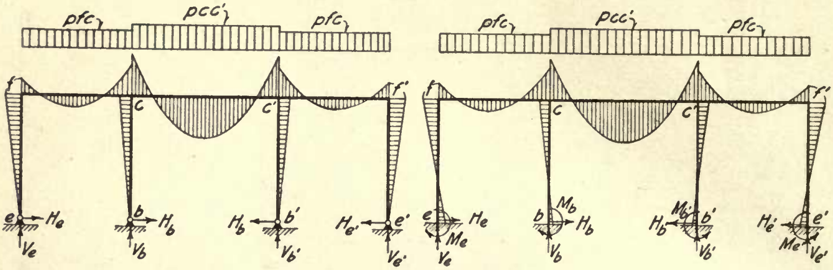


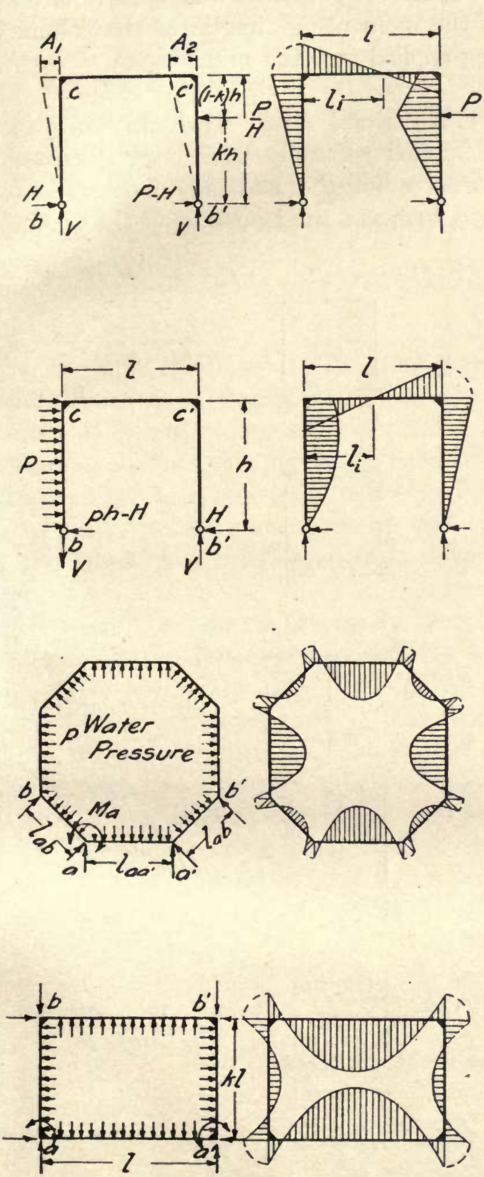
FIG. 20. SINGLE STORY THREE-SPAN FRAME WITH FOUR COLUMNS HAVING MIDDLE SPAN UNIFORMLY LOADED

TABLE 7

COEFFICIENTS OF BENDING MOMENT FOR SINGLE STORY THREE-SPAN FRAME HAVING FOUR COLUMNS

α = coefficient of pl^2 for bending moment at end of middle span.
β = coefficient of pl^2 for bending moment at top of intermediate column.

		$\frac{I_{cc}}{I_{bc}}$	$\frac{h}{l}$				
			0.50	0.75	1.00	1.25	1.50
Lower Ends of Columns Hinged	α	0.5	.0739	.0711	.0689	.0672	.0658
		1.0	.0689	.0658	.0634	.0620	.0605
		1.5	.0656	.0625	.0605	.0591	.0580
		2.0	.0634	.0605	.0587	.0573	.0565
		3.0	.0607	.0579	.0565	.0550	.0547
	β	0.5	.0563	.0488	.0431	.0387	.0351
		1.0	.0431	.0351	.0297	.0257	.0227
		1.5	.0351	.0276	.0227	.0194	.0169
		2.0	.0297	.0227	.0185	.0155	.0134
		3.0	.0228	.0169	.0134	.0112	.0096
Lower Ends of Columns Fixed	α	0.5	.0758	.0732	.0711	.0694	.0680
		1.0	.0711	.0680	.0658	.0640	.0626
		1.5	.0680	.0648	.0626	.0610	.0597
		2.0	.0658	.0626	.0605	.0592	.0580
		3.0	.0626	.0598	.0580	.0567	.0559
	β	0.5	.0610	.0542	.0488	.0444	.0408
		1.0	.0488	.0408	.0352	.0308	.0276
		1.5	.0408	.0330	.0276	.0238	.0210
		2.0	.0352	.0276	.0228	.0194	.0168
		3.0	.0276	.0210	.0168	.0142	.0122



$$H=\frac{k\left[3+(3-k^2)\frac{hI_{cc'}}{lI_{bc}}\right]}{2\left[3+2\frac{hI_{cc'}}{lI_{bc}}\right]}P$$

$$l_i=\frac{3+(3-k^2)\frac{hI_{cc'}}{lI_{bc}}}{2\left[3+2\frac{hI_{cc'}}{lI_{bc}}\right]}l$$

$$H=\frac{6+5\frac{hI_{cc'}}{lI_{bc}}}{3+2\frac{hI_{cc'}}{lI_{bc}}}\,\frac{ph}{8}$$

$$l_i=\frac{3}{4}\,\frac{2+\frac{hI_{cc'}}{lI_{bc}}}{3+2\frac{hI_{cc'}}{lI_{bc}}}\,l$$

$$M_a=-\frac{l_{aa'}^3+l_{ab}^3\frac{I_{aa'}}{I_{ab}}}{l_{aa'}+l_{ab}\frac{I_{aa'}}{I_{ab}}}\,\frac{p}{12}$$

$$M_a=-\frac{1+k^3\frac{I_{aa'}}{I_{ab}}}{1+k\frac{I_{aa'}}{I_{ab}}}\,\frac{pl^2}{12}$$

FIG. 21. FORMULAS FOR RECTANGULAR FRAME UNDER HORIZONTAL LOAD AND FOR OCTAGONAL AND RECTANGULAR RESERVOIRS

at the top of the middle columns and at the end of the middle span for the case in which the three spans are equal, the moments of inertia of the beams are equal, and the moments of inertia of the columns are equal, a uniform load being applied over the middle span.

13. *Square Frame under Horizontal Load.*—Hitherto only the cases in which the load was applied vertically have been discussed. It is frequently necessary to solve for the statically indeterminate stresses due to a horizontal force, such as a wind pressure or the braking

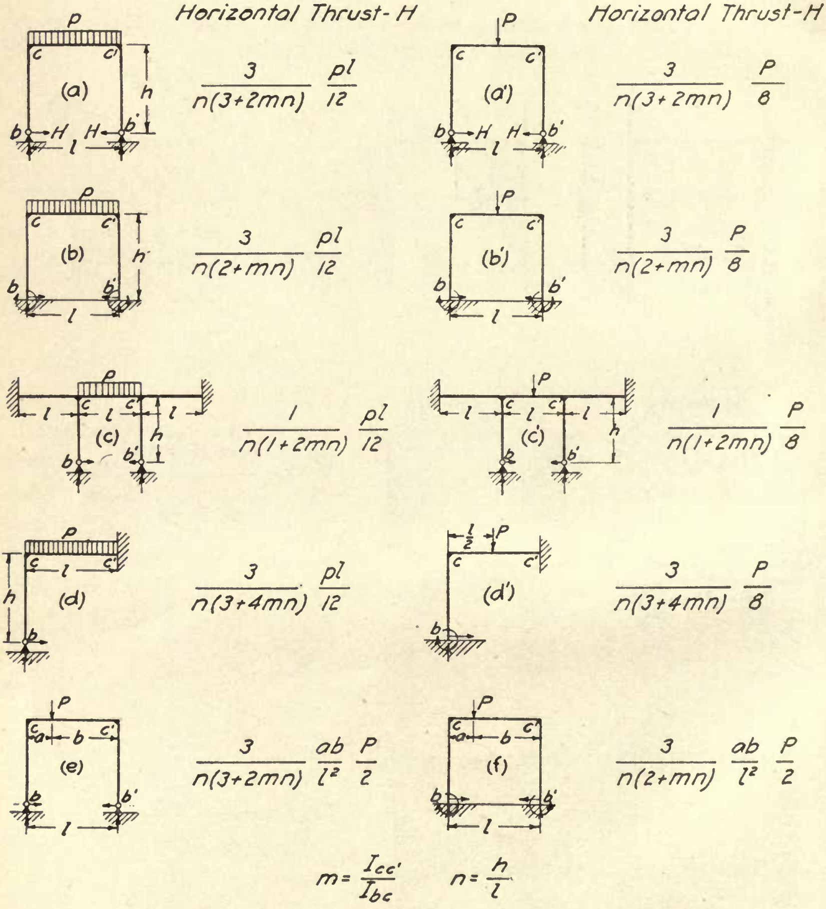

Horizontal Thrust- H
Horizontal Thrust-H
(a) $\frac{3}{n(3+2mn)}\frac{pl}{12}$
(a') $\frac{3}{n(3+2mn)}\frac{P}{8}$
(b) $\frac{3}{n(2+mn)}\frac{pl}{12}$
(b') $\frac{3}{n(2+mn)}\frac{P}{8}$
(c) $\frac{1}{n(1+2mn)}\frac{pl}{12}$
(c') $\frac{1}{n(1+2mn)}\frac{P}{8}$
(d) $\frac{3}{n(3+4mn)}\frac{pl}{12}$
(d') $\frac{3}{n(3+4mn)}\frac{P}{8}$
(e) $\frac{3}{n(3+2mn)}\frac{ab}{l^2}\frac{P}{2}$
(f) $\frac{3}{n(2+mn)}\frac{ab}{l^2}\frac{P}{2}$
p
P
c
c'
b
b'
H
h
l
a
b
$\frac{l}{2}$
$m=\frac{I_{cc'}}{I_{bc}}$
$n=\frac{h}{l}$

FIG. 22. HORIZONTAL REACTIONS FOR DISTRIBUTED LOADS AND CONCENTRATED LOADS

force of a locomotive. The method of determination of the statically unknowns is the same as that used for frames with vertical loads. A few cases have been taken as illustrations and the resulting equations are given in Fig. 21.

Another application is the water tank or reservoir subject to the static pressure of water, such as may be found in filter plants. Fig. 21 gives two examples of framed constructions of this character. It is seen that for a square tank having the same walls on the four sides the negative moment from the formula for rectangular tank becomes $\frac{1}{12}pl^2$ and the positive moment $\frac{1}{24}pl^2$, as is known from other sources.

14. *The Nature of the Resulting Formulas; Relation between Horizontal Reactions in Frame under Uniform Load and under Concentrated Load.*—It is interesting and important to note from the results of the foregoing analysis that there is a fixed relation between the horizontal reactions in the symmetrical frame under distributed loads and those in the same frame under concentrated loads. To show this relation a few cases have been selected as illustrative. These are shown in Fig. 22.

It has been stated previously that there is also a fixed relation between the horizontal thrust and the bending moment at the fixed column or beam ends, and the bending moment can be expressed in terms of the horizontal thrust. The bending moment at any section of a frame is a function of the horizontal thrust. Therefore, it may be stated that the statically indeterminate stresses in the *symmetrical* frame have a fixed relation under distributed and concentrated loads.

From the foregoing illustrations it will be seen that the horizontal thrusts at the column ends due to uniform and centrally concentrated loads may be expressed in the following forms:

Uniform load, $$H=K\frac{pl^2}{12}$$

Centrally concentrated load, $$H=K\frac{Pl}{8}$$

The coefficient K is the same in both cases, but varies with the form of frame. The formula for the horizontal thrust in the frame under concentrated load may be written directly if the formula for thrust in the frame under uniform load is known. An analysis of the statically indeterminate forces for a given case should first be made to find the form of the function.

The bending moment at the end of the span in these frames is:

For Case a, $$M=\frac{3}{3+2mn}\,\frac{pl^2}{12}=K_1\,\frac{pl^2}{12}$$

For Case a′, $$M=\frac{3}{3+2mn}\,\frac{Pl}{8}=K_1\,\frac{Pl}{8}$$

For Case b, $$M=\frac{2}{2+mn}\,\frac{pl^2}{12}=K_2\,\frac{pl^2}{12}$$

For Case b′, $$M=\frac{2}{2+mn}\,\frac{Pl}{8}=K_2\,\frac{Pl}{8}$$

It is known that when a beam is perfectly fixed at its ends the negative bending moments due to a distributed load and a centrally concentrated load are $\frac{pl^2}{12}$ and $\frac{Pl}{8}$, respectively. It is seen, therefore, that for these cases the bending moment at the end of the beam is obtained from the value of the end bending moment of a fixed beam by multiplying by K, a coefficient which depends upon the form of the frame, but is independent of whether the load is applied uniformly or is concentrated at the center of the span.

Returning to the nature of the formulas for the horizontal thrust at the lower column end of a frame, it is further seen from Fig. 22 that the given constant relation between the values of the horizontal thrusts for a frame under a distributed load and under a concentrated load still holds for the case in which a frame is subjected to a non-symmetrical load. These simple frames are sufficient to illustrate the general relation. It appears, therefore, that for the same frame the coefficient K remains constant and independent of the method of loading. This statement can easily be extended to the case of multiple concentrated loads, for then the horizontal thrust is the sum of the horizontal thrusts due to the individual concentrated loads. It will be found that this statement applies also to the non-symmetrical frames of Cases *d* and *d′*.

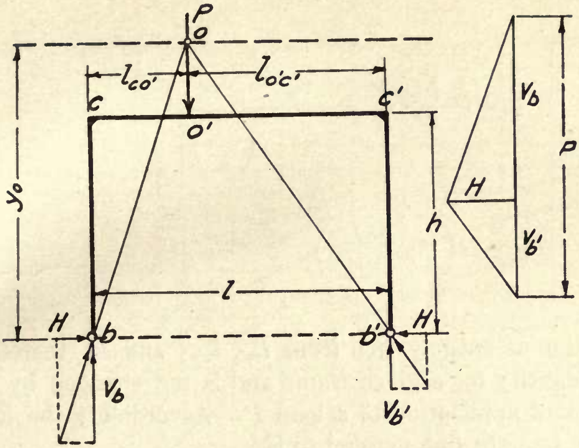


FIG. 23. LOCUS OF INTERSECTION OF REACTION LINES IN SINGLE SPAN SINGLE STORY FRAME UNDER SINGLE CONCENTRATED LOAD

For a concentrated load, it will be of interest to find the locus of y_o, the point of intersection of the lines of action of the reactions with the line of action of the load (see Fig. 23). In a complicated form of frame, there are, of course, many statically indeterminate quantities, but H is an important one. The remaining statically indeterminates have always the same factor in the denominator as H. Therefore, it is very interesting to know the form of the expression for H. It is evident that H (Fig. 23) is a function of l, h, I, and P in a given case.

Since the moments at b and b' (Fig. 23) are zero, the equilibrium polygon for the load P must pass through these points. Taking the moments of H and V_b about the point o

$$V_b l_{co'} - H y_o = 0 \qquad y_o = \frac{V_b l_{co'}}{H}$$

H and V_b are known in this case when P and $l_{co'}$ are given, and

$$H = \frac{\left(\frac{l}{h}\right)^2}{2\left(\frac{2I_{cc'}}{3I_{bc}} + \frac{l}{h}\right)} \cdot \frac{l_{co'} l_{o'c'}}{l^2} P$$

$$V_b = \frac{l_{o'c'}}{l} P$$

Therefore

$$y_o = \frac{l_{co} \cdot l_{o'c'}}{l} \frac{P}{H} = \frac{2\left(\frac{2I_{cc'}}{3I_{bc}} + \frac{l}{h}\right) l}{\left(\frac{l}{h}\right)^2}$$

or

$$y_o = 2h\left(1 + \frac{2hI_{cc'}}{3lI_{bc}}\right)$$

This equation is entirely free from $l_{co'}$, $l_{o'c'}$, and P; therefore, y_o is a constant quantity for a given frame and is not changed by the change of the point of application of a load P. Accordingly the locus of the point o is a straight line parallel to bb'.

In the case in which $\frac{I_{cc'}}{I_{bc}} = 1.0$ and $\frac{h}{l} = 1.0$, $y_o = \frac{10h}{3}$. For $\frac{I_{cc'}}{I_{bc}} = 2.0$ and $\frac{h}{l} = 1.0$, $y_o = \frac{14h}{3}$.

The equation for y_o permits the determination of the position of loads which gives the maximum reaction and stress in any member. The same method may be extended to any case, if it is remembered that when a column is fixed at its end the point of application of the reaction deviates from the neutral line of the column by $\frac{M_b}{V_b}$, where M_b is the end moment and V_b is the vertical reaction at that point.

15. *Effect of Variation in Moment of Inertia and Relative Height of Frame on Bending Moment in Horizontal Member.*—Fig. 24, 25, 26, 27, and 28 give bending moment coefficients for the beam of the central span for several cases of a three-span frame in which the spans are equal, the moments of inertia of the three beams are equal, and the moments of inertia of the columns are equal. The effect on the bending moment caused by variation in the relative values of the moments of inertia of members and in heights of frames is shown in these figures.

For a general comparison it is only necessary to consider the bending moment at the center of the span for load applied eccentrically with respect to the columns, since the effect on moments at other places

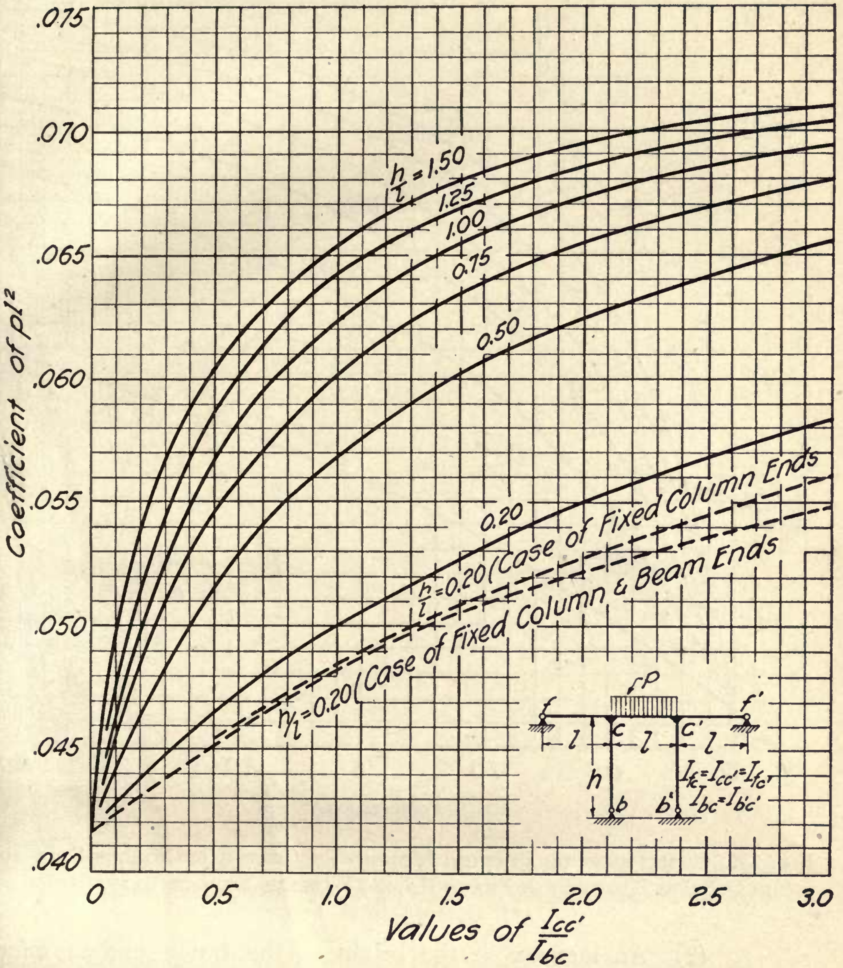


Fig. 24. Coefficient of Bending Moment at Center of Middle Span for Single Story Three-Span Frame Having Extreme Ends of Columns and Beams Hinged

and on thrusts will be similar. From the general nature of the curves shown the following conclusions are drawn:

(1) The bending moment is increased rapidly as the value of $\frac{I_{cc'}}{I_{bc}}$ increases from 0 to 1.5, but beyond that range the increase in bending moment is comparatively small.

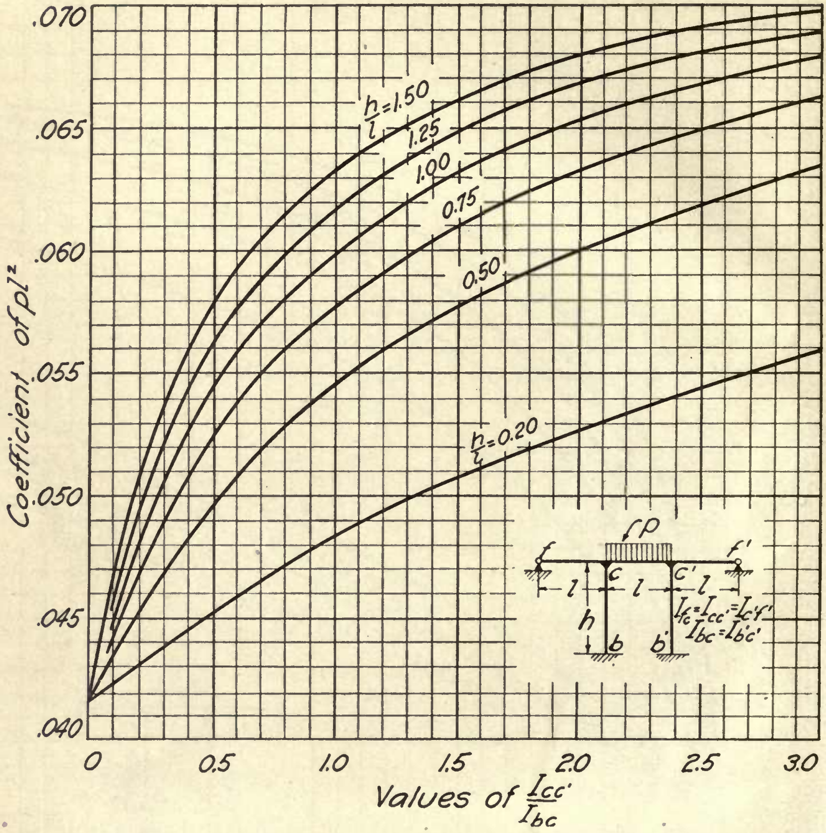


FIG. 25. COEFFICIENT OF BENDING MOMENT AT CENTER OF MIDDLE SPAN FOR SINGLE STORY THREE-SPAN FRAME HAVING EXTREME ENDS OF BEAMS HINGED AND COLUMN ENDS FIXED

(2) An increase in the height of the frame has an effect of the same nature on the bending moment as an increase in the ratio $\frac{I_{cc'}}{I_{bc}}$.

(3) The variation in coefficient of bending moment is wider in the frame hinged at ends of columns and beams than in the case of fixed ends.

(4) By the fixing of ends of columns and end beams the coefficient of positive bending moment is slightly decreased from that for hinged ends.

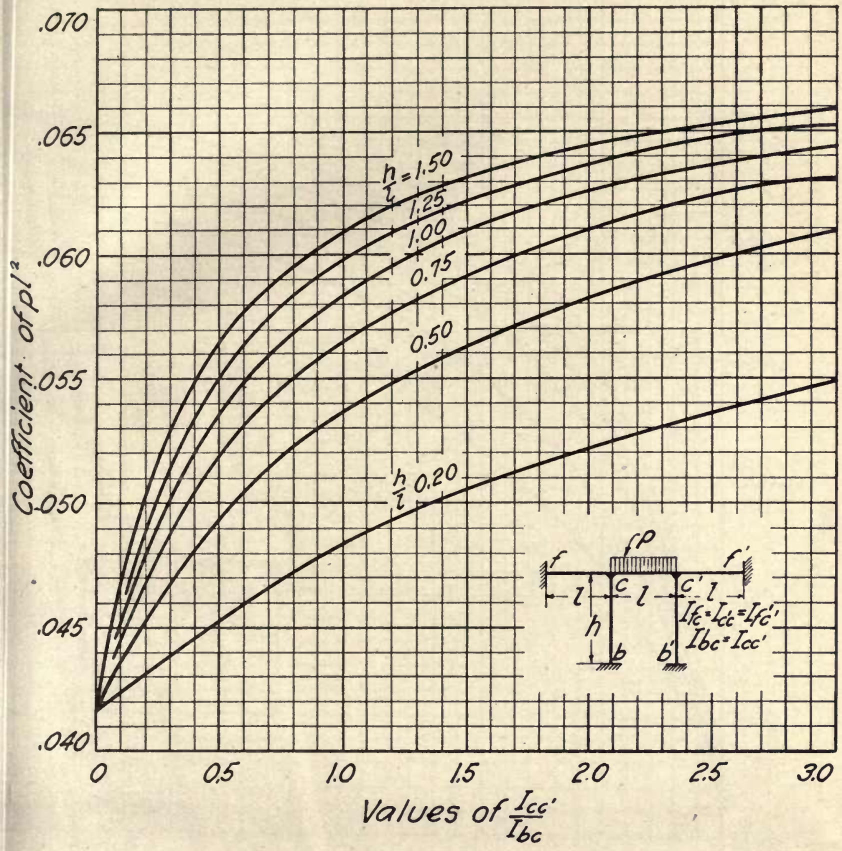


FIG. 26. COEFFICIENT OF BENDING MOMENT AT CENTER OF MIDDLE SPAN FOR SINGLE STORY THREE-SPAN FRAME HAVING ALL EXTERNAL CONNECTIONS FIXED

(5) In most common cases of panels under uniform load where the ratio $\frac{h}{l}$ is not far from 1.0 and $\frac{I_{cc'}}{I_{bc}}$ varies from 1.5 to 3.0, the bending moment at the center of the loaded span (case of equal spans) varies from about $\frac{1}{16}pl^2$ to about $\frac{1}{14}pl^2$, and may be conveniently assumed as $\frac{1}{15}pl^2$.

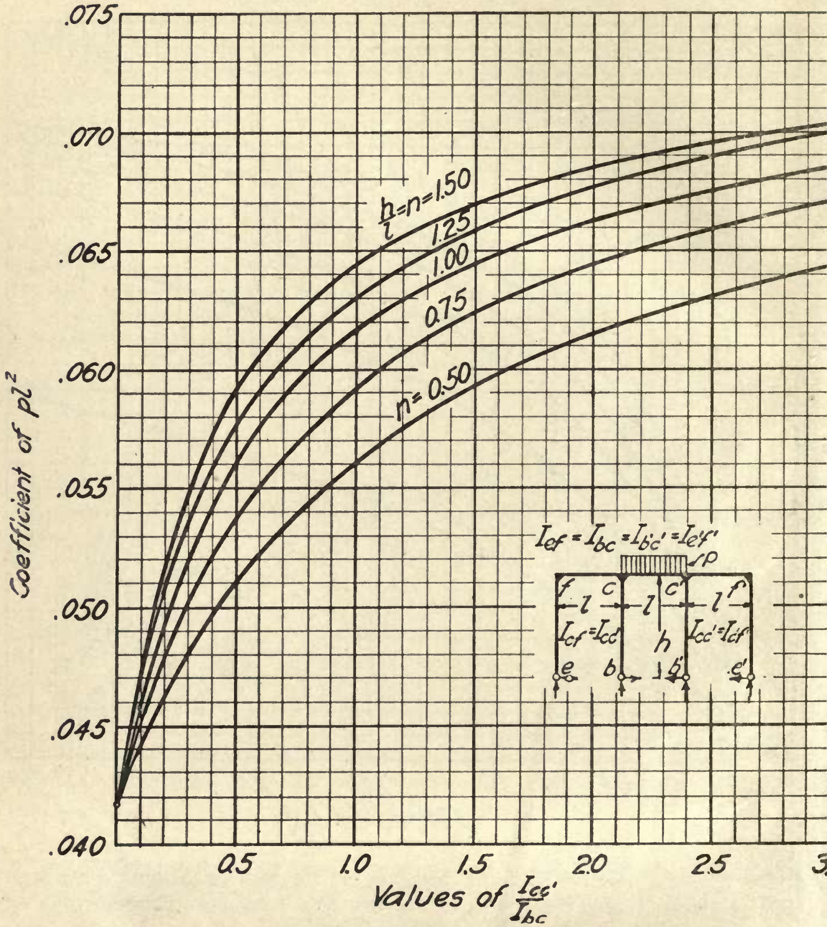


FIG. 27. COEFFICIENT OF BENDING MOMENT AT CENTER OF MIDDLE SPAN FO SINGLE STORY THREE-SPAN FRAME HAVING LOWER ENDS OF COLUMNS HINGED

16. *Effect of Variation in Moment of Inertia on Bending Momen in Vertical Member.*—The variation in bending moments in colum ends due to the variation in properties of the members for several case of a three-span frame is shown in Fig. 17 and 29 and in Tables 4, 6 and 7. In Table 6 the three spans are taken as equal and the stor height is taken equal to half the span.

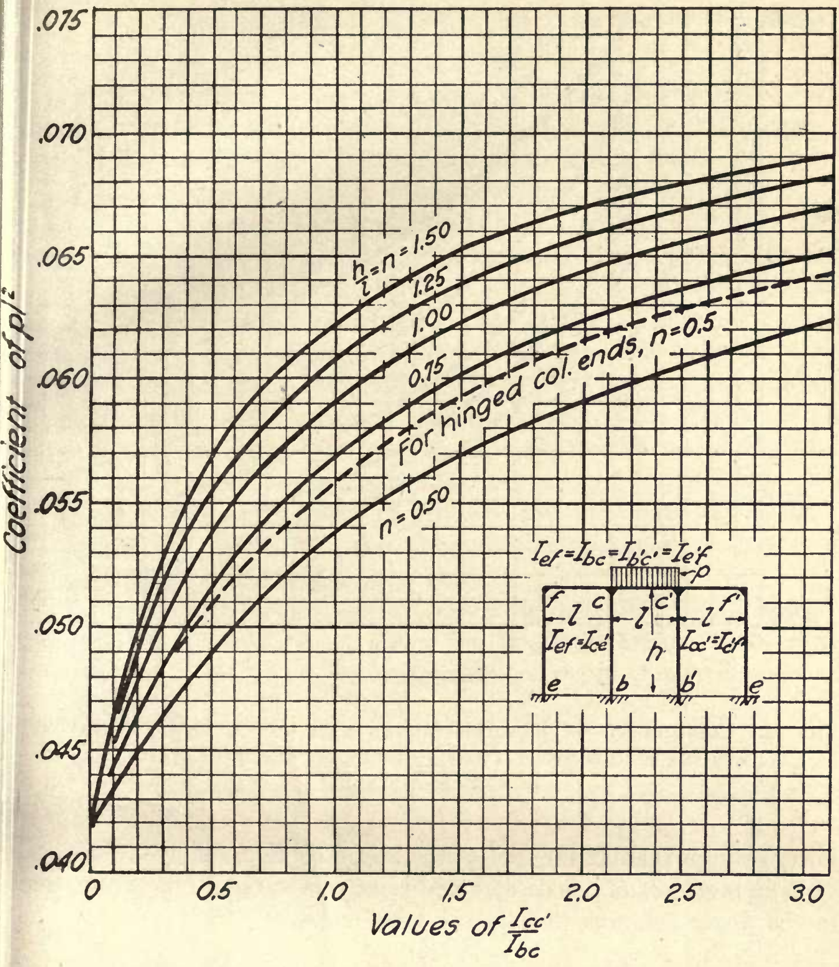


FIG. 28. COEFFICIENT OF BENDING MOMENT AT CENTER OF MIDDLE SPAN FOR SINGLE STORY THREE-SPAN FRAME HAVING LOWER ENDS OF COLUMNS FIXED

Values of coefficients of pl^2 for various values of $\frac{I_{cc'}}{I_{bc}}$ and $\frac{I_{cd}}{I_{bc}}$ are plotted in Fig. 17. It is seen from the diagram that for structures having the relations between span lengths and moments of inertia assumed in Table 6, higher bending stress will exist at the top of the

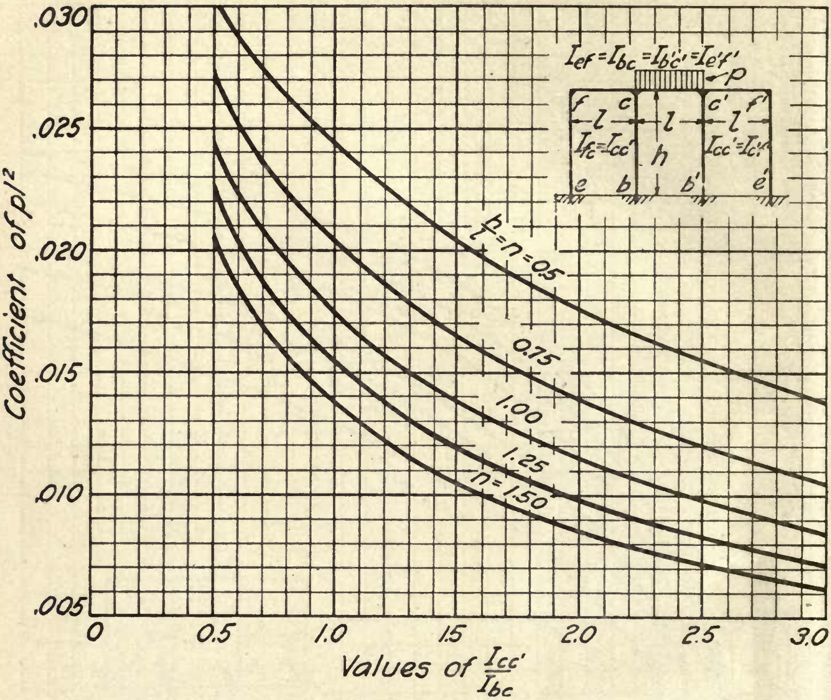


FIG. 29. COEFFICIENT OF BENDING MOMENT AT LOWER ENDS OF INTERIOR COLUMNS FOR SINGLE STORY THREE-SPAN FRAME HAVING LOWER ENDS OF COLUMNS FIXED

lower column than at the foot of the upper column and that the variations in moments of inertia assumed cause less variation in the moment in the upper columns than in the lower columns.

III. Tests on Rigidly Connected Reinforced Concrete Frames

17. *Test Specimens.*—Five types of frames were selected for the tests. The cross-section of the composing members varied from 8 by 8 in. to 8½ by 17⅜ in. The length of span of the frames was 6 ft. on centers except Frame No. 8, which had three spans of 4 ft. 8 in. The height of the frames varied from about 5 ft. to about 10 ft. The size and disposition of the reinforcing bars and the dimensions of the frames are shown in Fig. 30 to 34. Data of the frames are given in Table 8.

Care was taken in designing the test specimens to secure continuity of connected members and to obtain such proportions between moments of inertia and spans as would result in high bending stresses in the columns and beams at nearly the same time. The ends of the steel reinforcing bars were bent into hooks in the specimens having columns fixed at the ends. Bars continuous from one end to another were used for all frames. The radius of bends of the main rods was about 5 in. Several bars were welded and these welds were located at points where the bending moment was very small.

In the frames with stirrups, U-shaped or double U-shaped stirrups were used. They passed under the longitudinal bars and extended to the top of the beam. The size and spacing of the stirrups are given in Table 8.

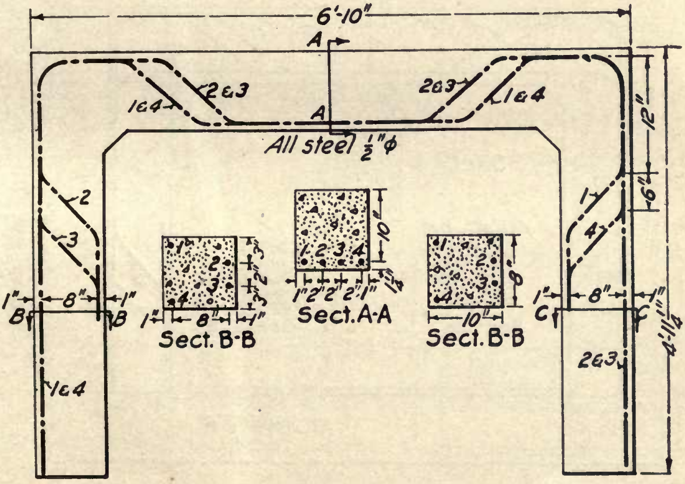


FIG. 30. SIZE AND DISTRIBUTION OF REINFORCING BARS IN FRAMES 1 AND 6

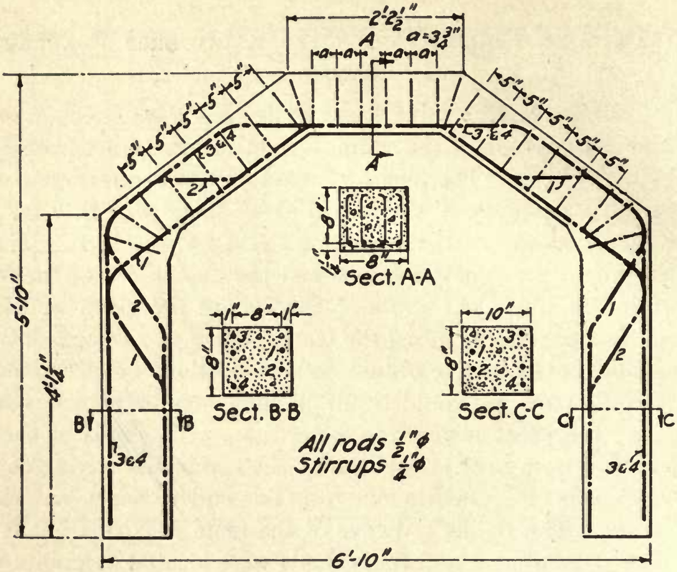


FIG. 31. SIZE AND DISTRIBUTION OF REINFORCING BARS IN FRAMES 2 AND 4

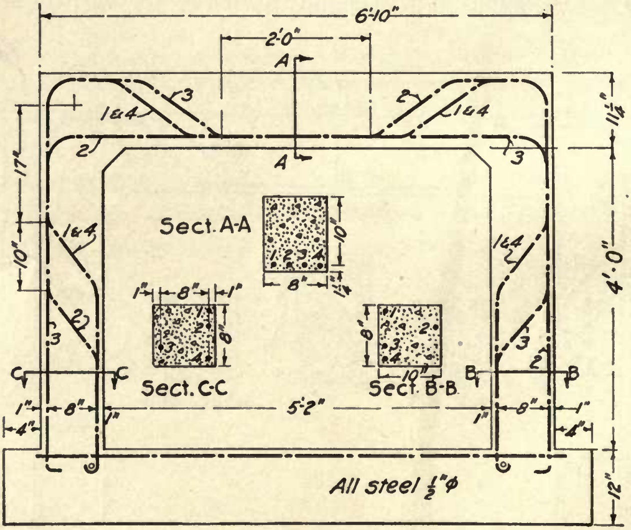


FIG. 32. SIZE AND DISTRIBUTION OF REINFORCING BARS IN FRAMES 3 AND 7

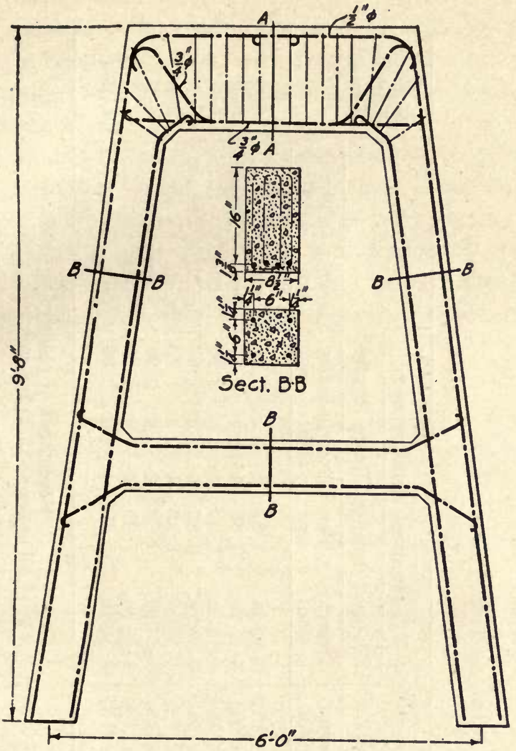


FIG. 33. SIZE AND DISTRIBUTION OF REINFORCING BARS IN FRAME 5

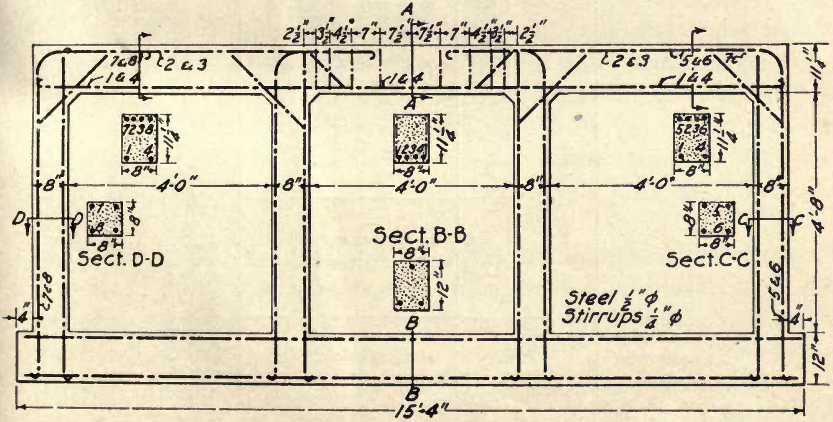


FIG. 34. SIZE AND DISTRIBUTION OF REINFORCING BARS IN FRAME 8

TABLE 8

DATA OF TEST FRAMES

Frame No.	Span Length c. to c.		Cross-Section		Longitudinal Reinforcement				Stirrups		Nominal Height of Frame ft.-in.
					Column (top)		Beam (center)				
	At Top ft.-in.	At Bottom ft.-in.	Column in. x in.	Beam in. x in.	Description	Per cent	Description	Per cent	Description	Spacing	
1	6–0	6–0	8 x10	8 x11¼	3–½ in.	0.82	4–½ in.	0.98	None	....	4–6
2	6–0	6–0	8 x10	8 x 9¼	3–½ in.	0.82	4–½ in.	1.23	¼–in.	3¾&5[1].	6–0
3	6–0	6–0	8 x10	8 x11¼	4–½ in.	1.09	4–½ in.	0.98	None	....	4–6
4	6–0	6–0	8 x10	8 x 9¼	4–½ in.	0.82	4–½ in.	1.23	¼–in.	3¾&5[1].	6–0
5	3–6	6–0	8½x 8½	8½x17⅜	4–½ in.	1.28	4–¾ in.	1.30	¼–in.	2½	9–0
6	6–0	6–0	8 x10	8 x11¼	3–½ in.	0.82	4–½ in.	0.98	None	....	4–6
7	6–0	6–0	8 x10	8 x11¼	4–½ in.	1.09	4–½ in.	0.98	None	....	4–6
8	4–8[2]	4–8[2]	8 x 8	8 x11¼	4–½ in.	1.40	4–½ in.	0.98	¼–in.	From 3½ to 7½[3]	5–2¼

All reinforcement of plain round bars.
All concrete of 1–2–4 mix.

---

[1] Double loop.
[2] Three equal spans.
[3] Only used in the middle span.

18. *Materials.*—The materials used in making the test frames were similar to those ordinarily used in reinforced concrete construction. The sand, stone, and cement were taken from the stock of the Laboratory of Applied Mechanics.

A good quality of crushed limestone ordered to pass over a ¼-inch sieve and through a 1-inch sieve was used. The sand was of good quality, hard, sharp, well-graded, and generally clean.

The reinforcing bars were plain round rods of open hearth mild steel. Test pieces were taken from the test frames after the test. Table 9 gives the results of the tension tests of the steel.

TABLE 9

TENSION TESTS OF REINFORCING STEEL

Nominal Diameter inches	Yield Point lb. per sq. in.	Ultimate Strength lb. per sq. in.	Per Cent Elongation in 8 in.
½	36 200	55 100	26.3
½	36 200	54 200	26.9
½	36 900	54 700	28.7
½	36 700	54 600	26.9
½	37 700	54 200	25.0
½	37 400	55 900	30.0
Average	36 850	54 783	27.3

Universal Portland cement was used for all specimens. Standard briquettes of neat cement gave an average tensile strength of 575 lb. per sq. in. at 7 days and 670 lb. per sq. in. at 28 days, and standard briquettes of 1–3 mortar 207 lb. per sq. in. at 7 days and 303 lb. per sq. in. at 28 days. Briquettes of 1–3 mortar made with the sand used in the concrete gave a strength of 279 lb. per sq. in. at 7 days and 353 lb. per sq. in. at 28 days. Tests with the Vicat needle indicated that initial set occurred in 3 hours and 15 minutes and final set in 6 hours.

Men skilled in this kind of work were employed in making the concrete. Care was taken in measuring, mixing, and tamping to secure concrete as nearly uniform as possible. All the concrete was made in the proportions, 1 part cement, 2 parts sand, and 4 parts stone, by volume. The mixing was done with a concrete mixing machine.

The results of compression tests on 6-in. cubes made from the concrete used in the frames are given in Table 10. Tests were made

TABLE 10

COMPRESSION TESTS OF CONCRETE CUBES AND CYLINDERS

Frame No.	Age at Test days	Maximum Load lb. per sq. in.		Frame No.	Age at Test days	Maximum Load lb. per sq. in.	
		6 in. Cube	8x16 in. Cyl.			6 in. Cube	8x16 in. Cyl.
1	64	1780	1150	5	61	3070	2670
1	64	1750		5	61	3100	
1	64	1680		5	61	2580	
Average		1740		Average		2920	
2	62	2210	1850	6	62	2605	2310
2	62	2250		6	62	2445	
2	62	2540		6	62	2510	
Average		2330		Average		2520	
3	73	2860	2050	7	60	2140	1970
3	73	2820		7	60	2390	
3	73	2840		7	60	2220	
Average		2840		Average		2250	
4	66	2600	1910	8	63	3288	3060
4	66	2580		8	63	3900	
4	66	2570		8	63	3653	
Average		2580		Average		3614	

on one 8 by 16-in. cylinder for each frame, and the axial deformation was measured to give a means of judging of the modulus of elasticity of the concrete used in the frames. Fig. 35 gives the stress-deformation diagrams for these cylinders. Table 10 gives the compressive strength of the cylinders.

19. *Making and Storage of Test Frames.*—It had been hoped to make the frames in a vertical position similar to that in practice, but because of the difficulty and added expense in doing this, all the frames were built directly on the concrete floor of the laboratory in a horizontal position with a strip of building paper beneath the forms.

The forms were generally removed after seven days, and the frames were lifted from the horizontal position after thirty days and were kept in a vertical position in the laboratory where they were made until the day they were tested. They were dampened every morning for two weeks after making to prevent too rapid drying, and were dampened occasionally after that time. The temperature of the room ranged from 55 to 70 degrees F.

20. *Testing.*—To develop high stresses in the beam and in the columns nearly at the same time one-third point loadings were used for many of the frames. In Frame 5 the centrally concentrated load was used to develop as high a flexural stress in the columns as possible.

In Frame 8 in order to see the effect of the eccentric load on the adjacent spans and at the same time to produce high bending stresses in the middle beam and in the central columns, a uniform load on the middle span was selected, the load being applied through a number of spiral springs.

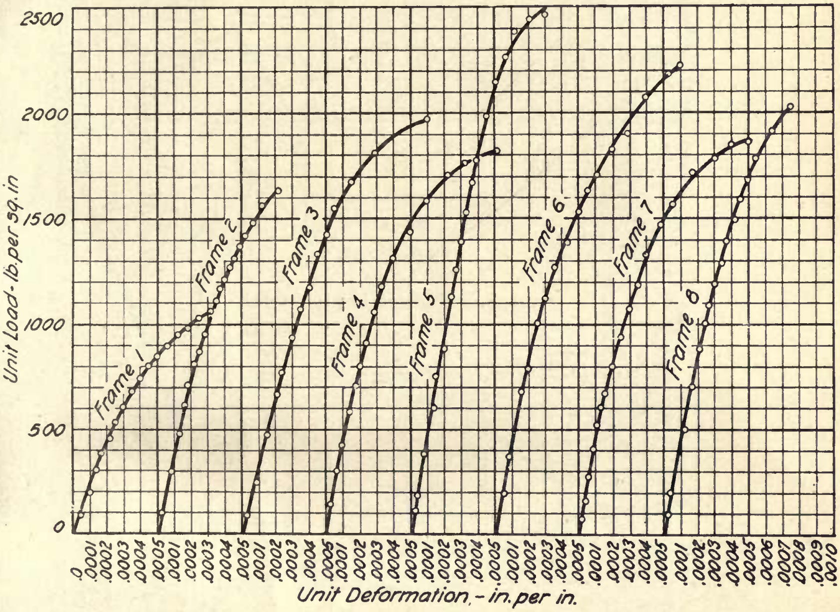


FIG. 35. STRESS-DEFORMATION DIAGRAMS FOR CYLINDERS

The positions of the loads for the different frames are shown in Fig. 36 to 43. The specimens were tested in the 600,000-lb. Riehle testing machine in the Laboratory of Applied Mechanics of the University of Illinois. Deflections were read on some of the frames. The deformations of the steel and of the concrete were measured at the various parts of the frames for each load applied.

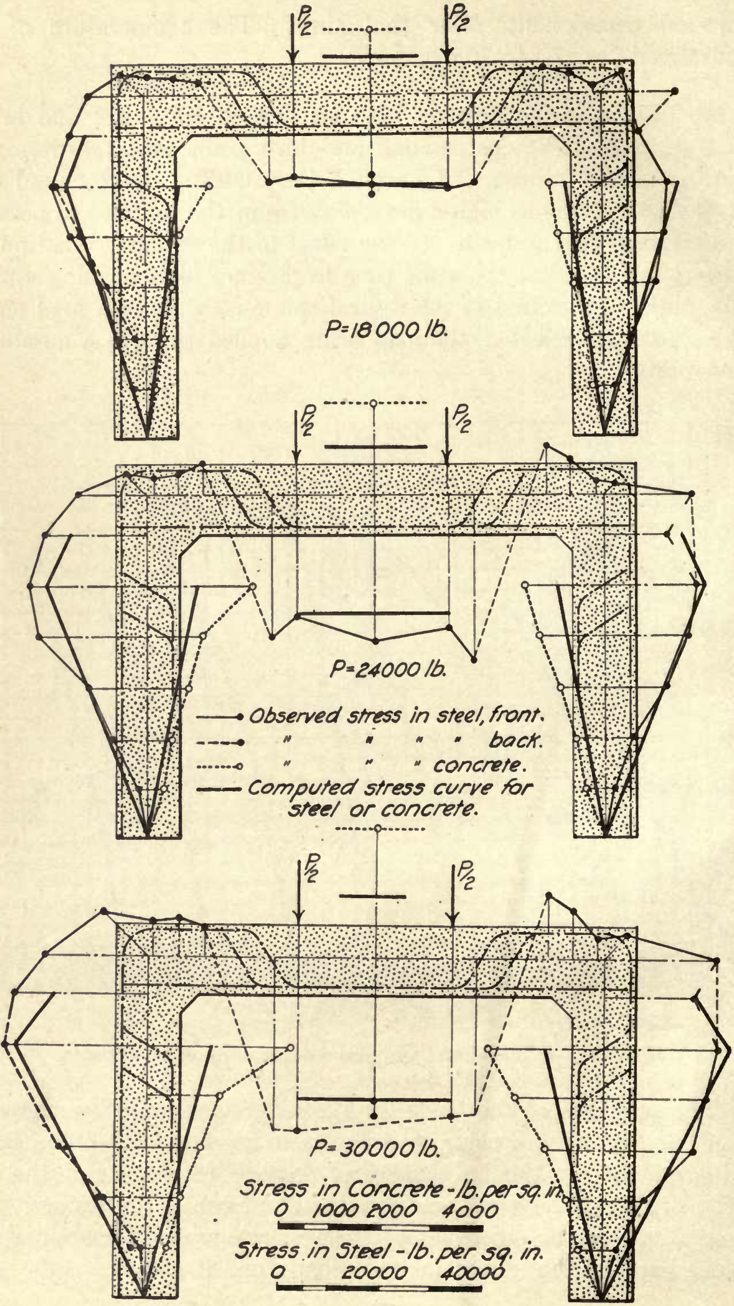


FIG. 36. OBSERVED AND COMPUTED STRESSES IN FRAME 1

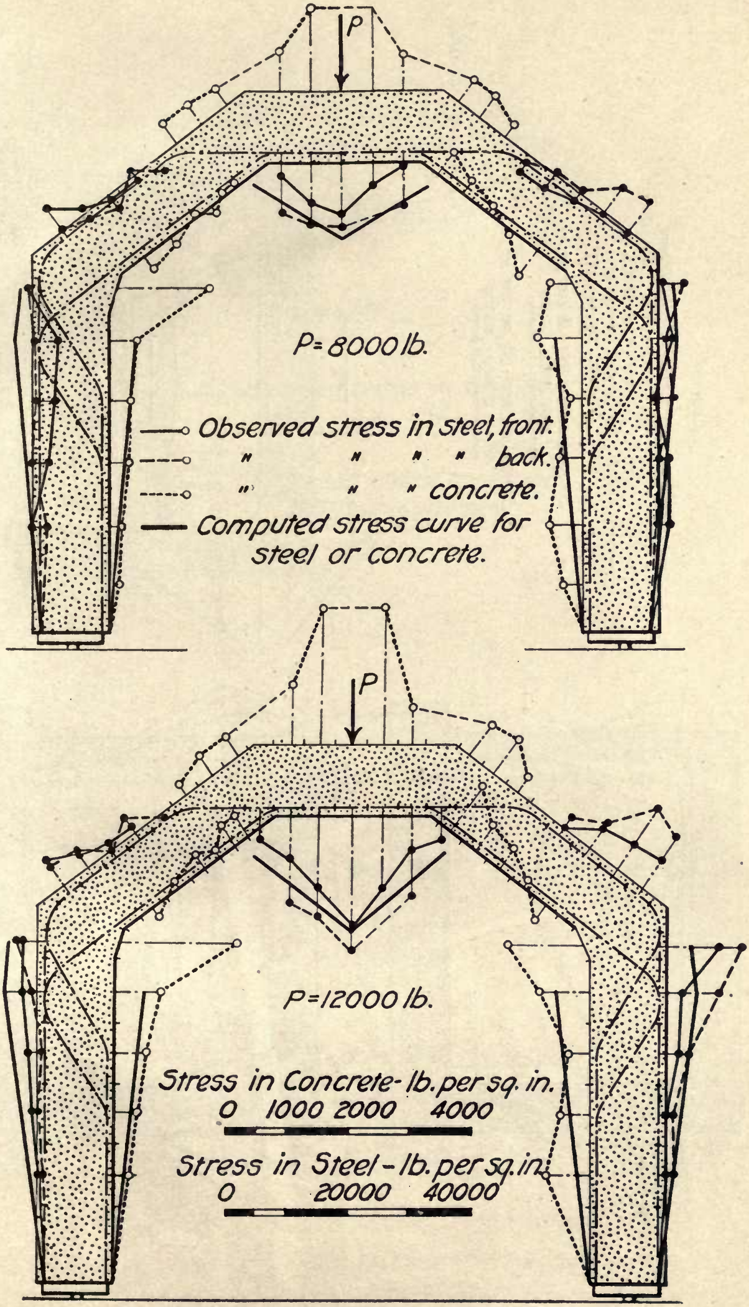

P
P= 8000 lb.
Observed stress in steel, front.
" " " " back.
" " " concrete.
Computed stress curve for steel or concrete.
P
P=12000 lb.
Stress in Concrete - lb. per sq. in.
0 1000 2000 4000
Stress in Steel - lb. per sq. in.
0 20000 40000

Fig. 37. Observed and Computed Stresses in Frame 2

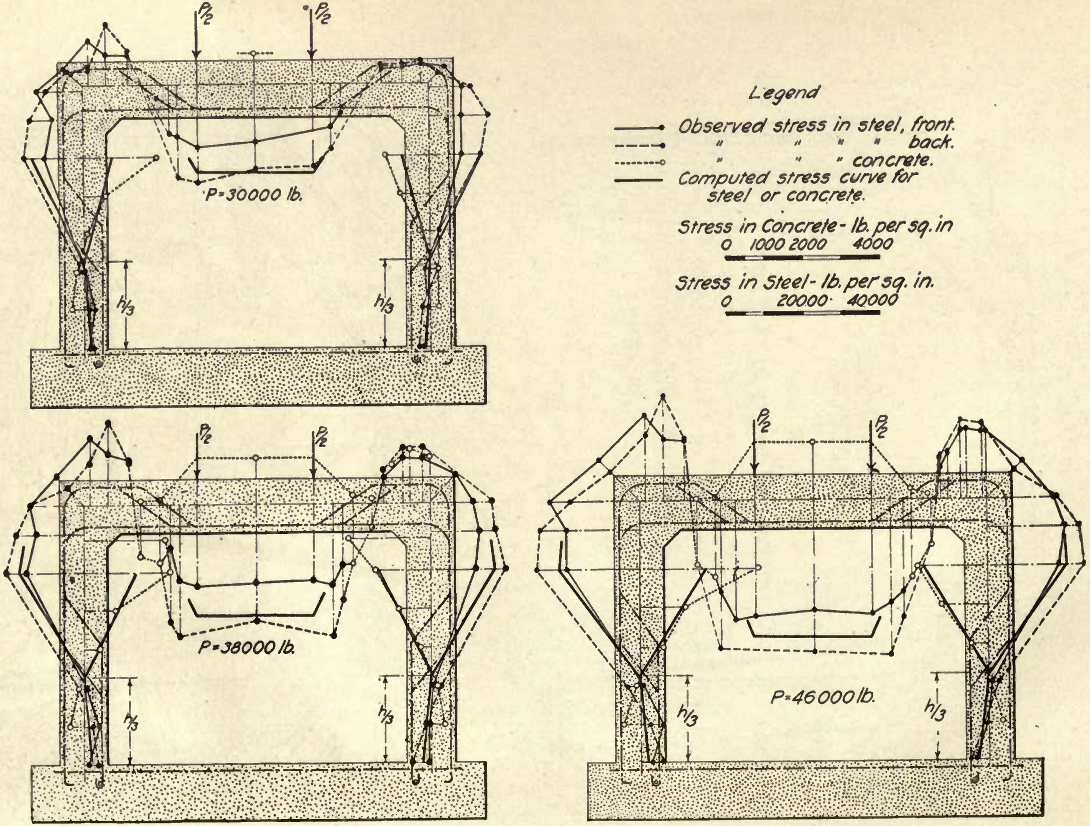


FIG. 38. OBSERVED AND COMPUTED STRESSES IN FRAME 3

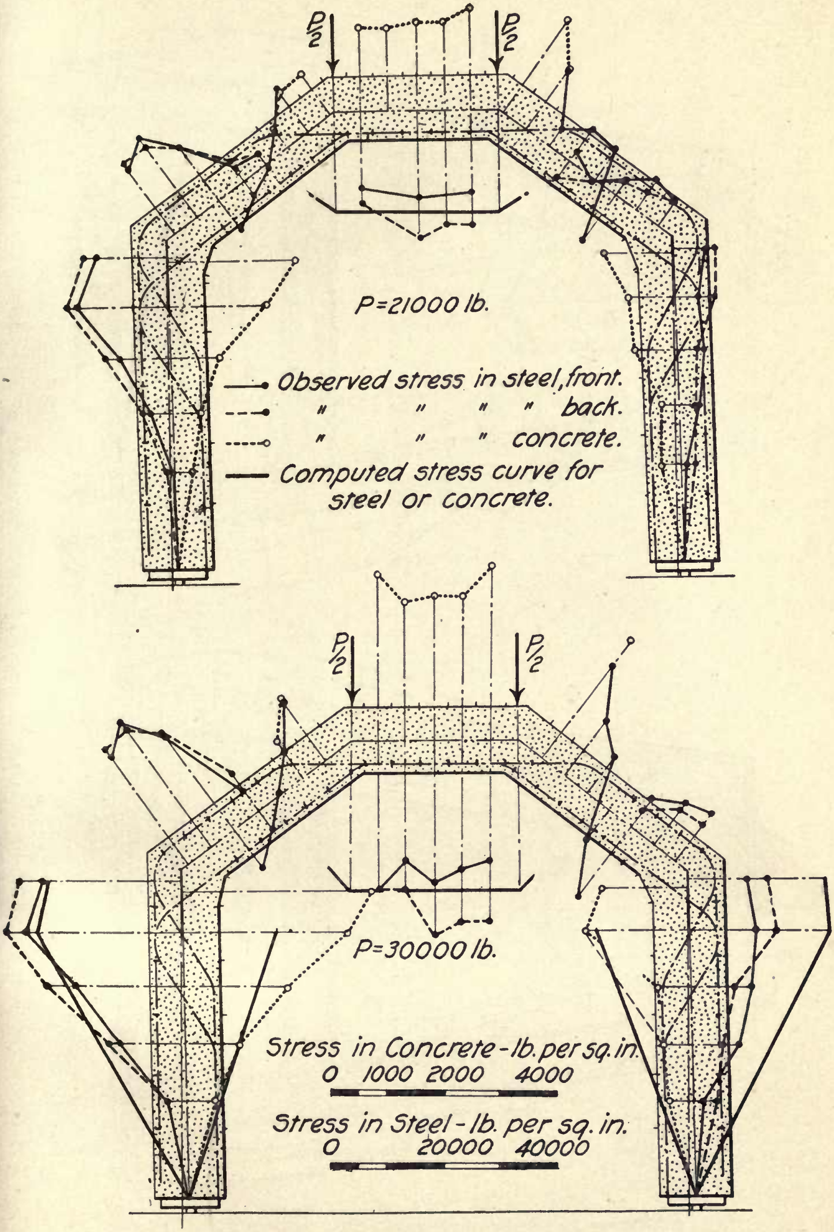

P/2
P/2
P=21000 lb.
Observed stress in steel, front.
" " " " back.
" " " concrete.
Computed stress curve for steel or concrete.
P/2
P/2
P=30000 lb.
Stress in Concrete - lb. per sq. in.
0 1000 2000 4000
Stress in Steel - lb. per sq. in.
0 20000 40000

Fig. 39. Observed and Computed Stresses in Frame 4

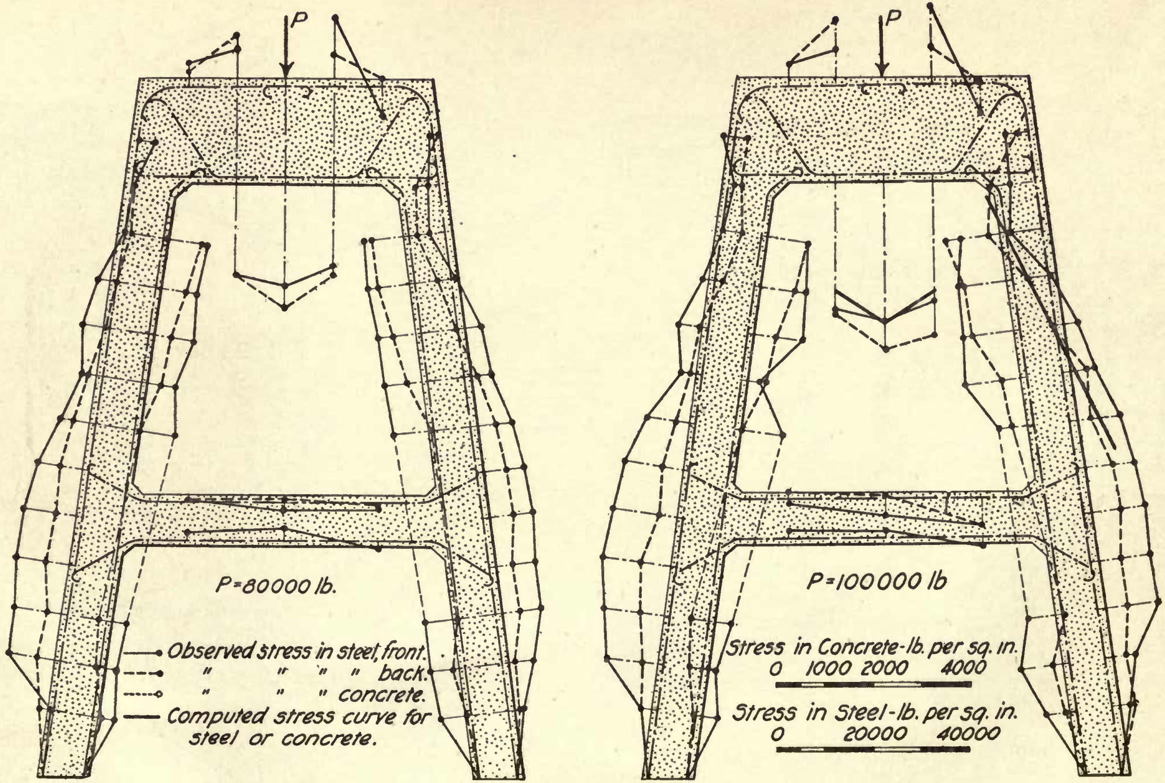


FIG. 40. OBSERVED AND COMPUTED STRESSES IN FRAME 5

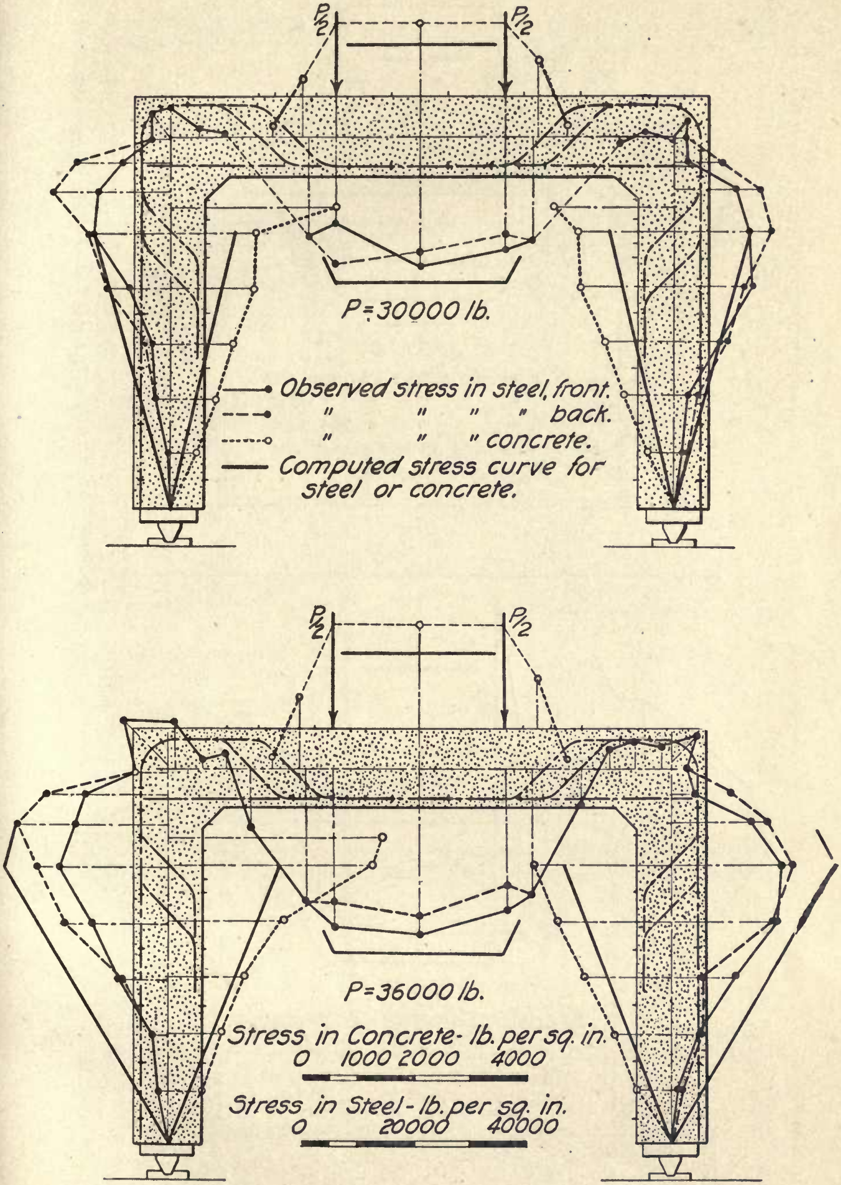

P/2
P/2
P=30000 lb.
Observed stress in steel, front.
" " " " back.
" " " concrete.
Computed stress curve for steel or concrete.
P/2
P/2
P=36000 lb.
Stress in Concrete - lb. per sq. in.
0 1000 2000 4000
Stress in Steel - lb. per sq. in.
0 20000 40000

FIG. 41. OBSERVED AND COMPUTED STRESSES IN FRAME 6

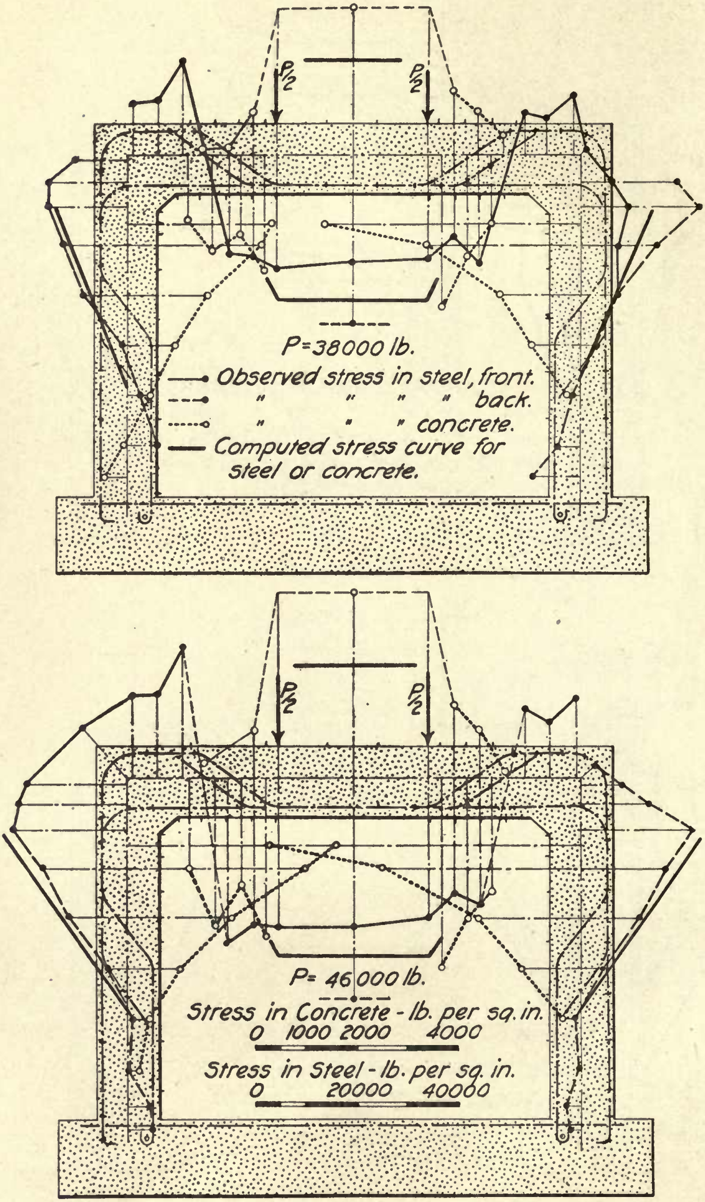

P/2
P/2
P=38000 lb.
Observed stress in steel, front.
" " " " back.
" " " concrete.
Computed stress curve for steel or concrete.
P/2
P/2
P= 46000 lb.
Stress in Concrete - lb. per sq. in.
0 1000 2000 4000
Stress in Steel - lb. per sq. in.
0 20000 40000

FIG. 42. OBSERVED AND COMPUTED STRESSES IN FRAME 7

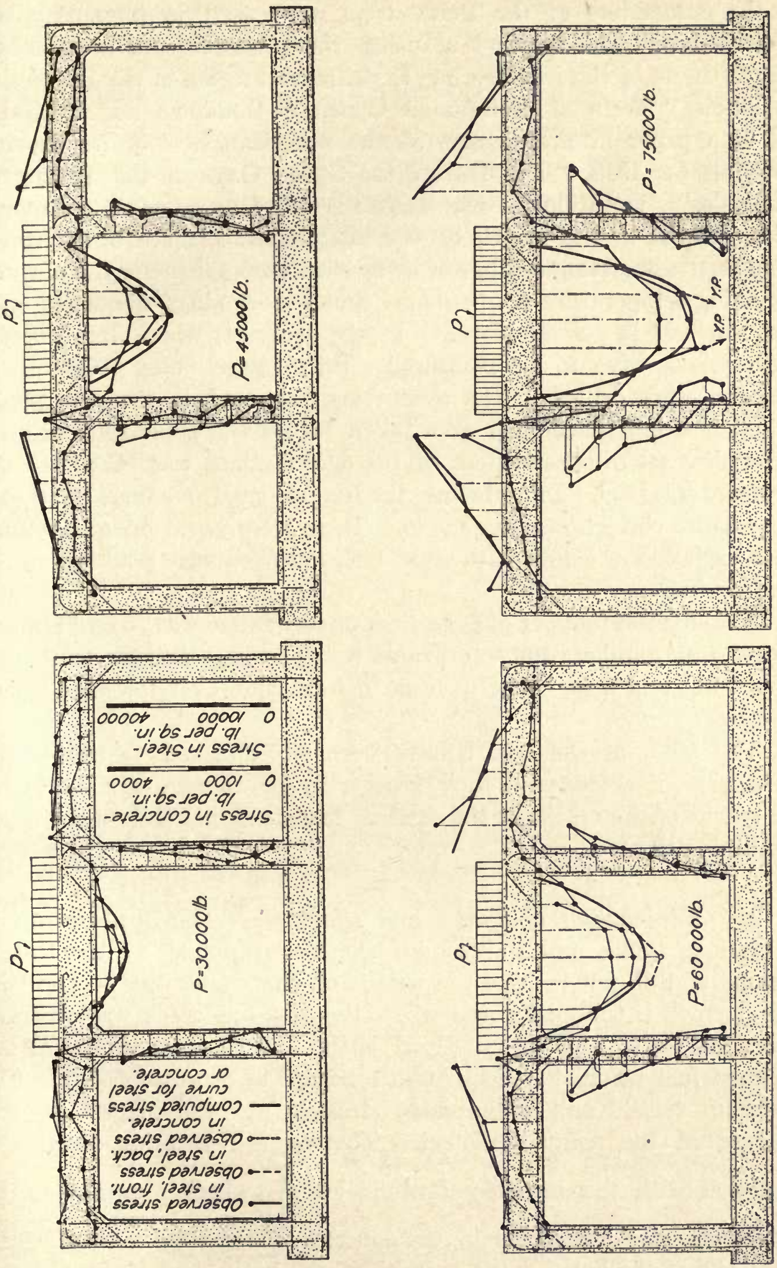


FIG. 43. OBSERVED AND COMPUTED STRESSES IN FRAME 8

Extensometers of the Berry type were used in measuring the deformations. The method of using these instruments is described in Bulletin 64 of the Engineering Experiment Station of the University of Illinois, "Tests of Reinforced Concrete Buildings under Load," and in a paper in Proceedings of the American Society for Testing Materials for 1913, "The Use of the Strain Gage in the Testing of Materials." Variation in temperature is sufficient to cause an appreciable change in the length of the instrument. Hence observations on an unstressed standard bar of invar steel were taken for the purpose of making temperature corrections. Small steel plugs, about one inch long, were set in plaster of paris in the concrete, where the concrete deformations were to be measured. Small gage holes, 0.055 in. in diameter, were drilled in the reinforcing bars and in the steel plugs. Two sets of initial readings were taken before the application of load. A complete set of observations of the deformations was taken at each increment of load. In reducing the strain gage readings to stress, temperature corrections were made. These were based on an assumed linear variation of length with time between successive readings on the standard bar.

The smallest number of gage lines on any frame was 75 on Frame 1; the greatest number, 163, on Frame 8. The gage lengths used were 2 in., 4 in., and 8 in. The average deformation over the gage length was used.

The faces of the test frames were whitewashed to enable the appearance of cracks and their growth to be more easily observed. The extent of the cracks at the several loads was marked on the specimen during the test.

21. *Explanation of Tables and Diagrams.*—The loads given in the various tables and figures are the loads applied by the testing machine and do not include the weight of the frame itself. The load at first crack is the load noted when the first fine crack was observed during the test. The ultimate load is the highest load applied to the specimen just before the load carried began to decrease slowly. The maximum tensile and compressive stresses are the highest stresses observed at the points specified. The vertical shearing stress was calculated with the ordinary formula $v=\frac{V}{bjd}$, where v represents the vertical shearing unit-stress in the concrete, V the total vertical shear at the end of the beam, b the breadth of the beam, and jd the distance

from the center of the steel to the center of the compressive stresses in the concrete. The bond stress in the beam was computed by means of the formula $u=\frac{V}{mojd}$, where u is the bond stress per unit of area of the surface of the reinforcing steel, m the number of reinforcing bars, and o the circumference or periphery of one reinforcing bar. The values of jd were selected with reference to the amount of reinforcing steel and the modulus of elasticity of the concrete.

Loads are given in pounds, unit-stresses in pounds per square inch, and moments in inch-pounds.

TABLE 11

VALUES OF MODULUS OF ELASTICITY OF CONCRETE USED IN STRESS COMPUTATIONS

Frame	E	Frame	E
1	2 100 000	5	3 000 000
2	3 600 000	6	3 900 000
3	2 070 000	7	3 700 000
4	3 600 000	8	3 300 000

The so-called observed stresses have been obtained from the observed deformations by using a modulus of elasticity of 30,000,000 lb. per sq. in. for the steel, and for the concrete the values given in Table 11.

Table 12 contains general data of the tests of the frames.

Table 13 gives computed stresses at the three points in each frame which are shown in Fig. 44, calculated by means of the formulas given in the preceding pages, the values being expressed in terms of the load applied to the frame. The values in columns marked I were obtained on the assumption that the concrete has full tensile strength; those in columns marked II on the assumption of no tensile strength. The division of the direct stress between concrete and steel was computed by the usual formulas for reinforced concrete columns.

Other explanations of tables and diagrams are made elsewhere.

22. *Phenomena of Frame Tests.*—As may be expected in reinforced concrete flexural members, the tensile stresses in the steel were very small at low loads. Undoubtedly this effect was largely due to the ability of the concrete to carry tensile stress. As soon as the con-

TABLE 12

GENERAL DATA OF TESTS OF FRAMES

Frame No.	Age days	Method of Loading	Load at First Crack pounds	Maximum Load at which Deformation was measured pounds	Ultimate Load at Failure pounds	Maximum Stress pounds per square inch							
						Tensile Stress in Steel				Compressive Stress in Concrete		Vertical Shearing Stress in Top Beam	Bond Stress in Top Beam
						Center of Top Beam	Ends of Top Beam	Bent up Bar in Beam	In Column	Center of Top Beam	In Column		
1	63	Concentrated Loads at Third Point	12 000	36 000	40 500	36 000	16 000	21 700	35 200	3 320[1]	3 130[1]	266	333
2	62	Concentrated Load at Center	8 000	14 000	....	32 900	28 900	......	12 300	3 840[1]	1 570	124	159
3	63	Concentrated Loads at Third Point	21 000	46 000	61 000	39 500	20 300	15 700	23 800	1 500	3 050	326	415
4	63	Concentrated Loads at Third Point	10 000	30 000	50 000	25 300	21 800	......	27 600	2 590	2 920	267	340

TABLE 12—(CONTINUED)

GENERAL DATA OF TESTS OF FRAMES

Frame No.	Age days	Method of Loading	Load at First Crack pounds	Maximum Load at which Deformation was measured pounds	Ultimate Load at Failure pounds	Maximum Stress pounds per square inch							
						Tensile Stress in Steel				Compressive Stress in Concrete		Vertical Shearing Stress in Top Beam	Bond Stress in Top Beam
						Center of Top Beam	Ends of Top Beam	Bent up Bar in Beam	In Column	Center of Top Beam	In Column		
5	61	Concentrated Load at Center	40 000	100 000	146 000	36 400	5 700	......	11 800	....	12 200[2]	425	383
6	58	Concentrated Loads at Third Point	18 000	36 000	46 000	29 800	7 400	10 300	27 000	2 750	3 840[1]	253	323
7	62	Concentrated Loads at Third Point	21 000	46 000	61 000	44 400	25 500	18 100	30 200	3 700[1]	4 130[1]	323	413
8	61	Uniform Load over Middle Span	45 000	60 000[3]	134 000	30 900[1]	13 200	......	14 700	2 540[4]	14 700[2]	425	540

[1] Not reliable, elastic limit exceeded.

[2] Steel stress in compression side of column.

[3] Not maximum load.

[4] Stress at point one inch distant from extreme fiber.

TABLE 13

COMPUTED STRESSES FOR TEST FRAMES IN TERMS OF TOTAL APPLIED LOAD

No. of Frame	H	Point	M inch unit	p	Direct Stress		I			II		
					Steel	Concrete	k	f_s	f_c	k	f_s	f_c
1	0.082 P	A	—3.43 P	.0068	.070 P	.0049 P				0.35	0.82 P	.039 P
		B	—4.04 P	.0109	.070 P	.0049 P				0.41	0.60 P	.041 P
		C	+7.43 P	.0098	.012 P	.0009 P				0.41	1.09 P	.054 P
2	0.121 P	A	—4.84 P	.0082	.049 P	.0059 P	0.58	0.15 P	.041 P	0.31	0.97 P	.059 P
		B	+5.28 P	.0123	.015 P	.0018 P				0.36	0.93 P	.067 P
		C	+9.78 P	.0123	.015 P	.0018 P				0.36	1.75 P	.123 P
3	0.145 P	A	+1.37 P	p .0054 p ′ = .0027	.053 P	.0059 P	0.52	0.032 P	.016 P			
		B	—3.02 P	.0109	.052 P	.0058 P	0.54	0.11 P	.029 P	0.35	0.44 P	.036 P
		C	+5.40 P	.0098	.015 P	.0017 P	0.60	0.17 P	.032 P	0.35	0.76 P	.045 P
4	0.100 P	A	—4.30 P	.0082	.049 P	.0059 P	0.58	0.15 P	.041 P	0.31	0.85 P	.054 P
		B	+4.90 P	.0123	.013 P	.0015 P	0.61	0.20 P	.034 P	0.36	0.87 P	.063 P
		C	+4.90 P	.0123	.013 P	.0015 P				0.36	0.87 P	.063 P
5		A	0.23 P	p=p′=.0064	.060 P		0.50	0.047 P	.074 P^1			
		A_1	0.10 P	p=p′=.0064	.060 P		0.50	0.055 P	.067 P^1			
		B	0.62 P	p=p′=.0064	.060 P		0.50	0.024 P	.097 P^1	0.242	0.19 P	.066 P
		C	8.40 P	.013	.003 P		0.57	0.150 P		0.42	0.30 P	
6	0.077 P	A	—3.42 P	.0068	.040 P	.0052 P	0.58	0.10 P	.030 P	0.28	0.81 P	.047 P
		B	—4.04 P	.0109	.040 P	.0052 P	0.59	0.12 P	.033 P	0.34	0.60 P	.046 P
		C	+7.48 P	.0098	.007 P	.0009 P	0.59	0.22 P	.042 P	0.33	1.06 P	.065 P

TABLE 13—(CONTINUED)

COMPUTED STRESSES FOR TEST FRAMES IN TERMS OF TOTAL APPLIED LOAD

No. of Frame	H	Point	M inch unit	p	Direct Stress		I			II		
					Steel	Concrete	k	f_s	f_c	k	f_s	f_c
7	0.145 P	A	2.03 P	$p = .0054$ $p' = .0027$	.048 P	.0059 P	0.52	$f_s = 0.028P$ $f'_s = 0.178P$	.024 P			
		B	—3.02 P	.0109	.047 P	.0058 P	0.54	0.13 P	.029 P	0.35	0.43 P	.038 P
		C	+5.40 P	.0098	.014 P	.0017 P				0.35	0.76 P	.048 P
8	I II											
	H_e .009 P .010 P	A		.0098	Neglected		0.33	0.46 P	.028 P	0.33	0.51 P	.032 P
	H_b .003 P .023 P	A_1		$p = .0073$ $p' = .0049$	Neglected		0.58	0.47 P	.007 P	0.28	0.24 P[2] 0.34 P[3]	.0102 P[2] .0146 P[3]
	V_e .047 P .039 P	B		$p = p' = .0062$	.070 P		0.50	0.166 P	.216 P[1]			
	V_b .547 P .539 P	C		$p = p' = .0049$	Neglected		0.50	0.06 P	.06 P[1]	0.24	0.31 P	.047 P[1]

---

[1] Compressive stress in steel, f'_s

[2] Same conditions as I except central beam is considered as broken in tension.

[3] Same conditions as I except beams and intermediate columns are considered as broken in tension.

crete on the tension side of the member was sufficiently stretched, a vertical tension crack formed on the beam underneath the load and then a crack formed at the side near the juncture of the column and the beam, in most cases. After the formation of these cracks, the tension in these parts was taken mainly by the reinforcing bars. As the loads were increased the cracks developed and new cracks appeared on the tension side between the points of application of the load on the beam, and horizontal cracks formed at regular intervals in the columns.

The tensile stress due to the negative bending moment within the space occupied by the juncture of the beam and the column was small, and in these places tension cracks did not form in many frames until high loads were applied. The bent-up bars in the beam came into action as soon as tension cracks formed in their vicinity, and in the bent-up portions tensile stresses as high as 22,000 pounds per square inch were developed in several instances. The tensile stresses in the steel at the fixed ends of the columns were rather low. The tensile strength of the concrete in this part apparently reduced the tensile stress in the steel.

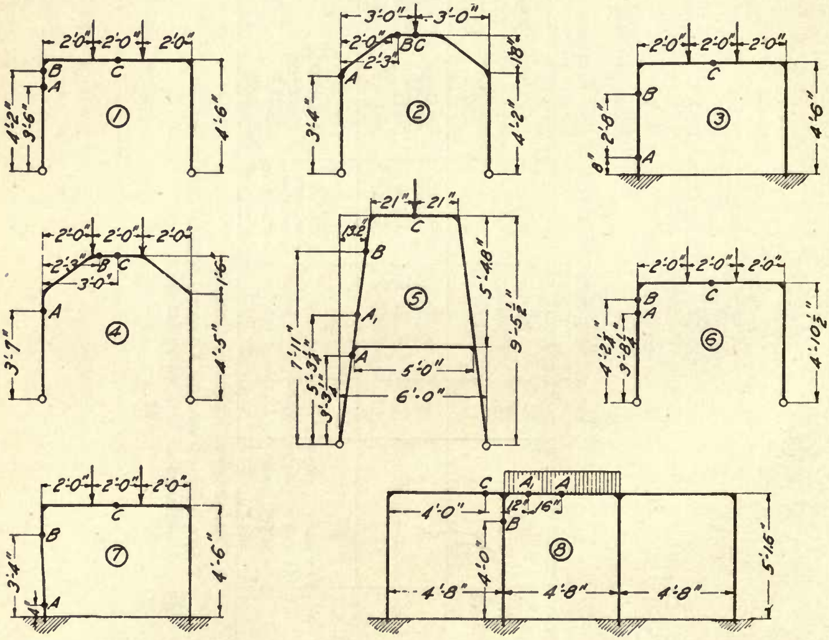


FIG. 44. POINTS USED FOR COMPARISON OF COMPUTED AND OBSERVED STRESSES

High compressive stresses were developed in the concrete in the upper portion of the columns below the intersection with the beam, and the maximum compression was observed along the sharp corner at the juncture of the beam and the column, as might be expected. This is due to the curved beam action at the rigid joint. In each frame the maximum load was higher than the load expected.

Views of the frames which show the location of cracks are given in Fig. 49 to 57.

The general phenomena of the tests of the individual frames are given in the following brief notes.

Frame 1—*Square Frame with Columns Hinged at Lower End.*—Nominal span length was 6 feet. Total height was 5 ft. 2 in. Frame was loaded at the one-third points of the span of the horizontal member. Fig. 49 shows the frame in the testing machine. The location of the cracks is shown in Fig. 51. At the 12,000-lb. load the first fine crack in the beam appeared directly under a load point and extended from the bottom of the member vertically 2 in. to the level of the reinforcement. At the same load the first noticeable cracks appeared in the columns, one in the outside edge of the column on a level with the bottom surface of the beam and one at 2 ft. 5 in. from the bottom of each column end. No crack appeared in the top side of the beam until the load was increased to 36,000 lb. At that load cracks appeared 8 in. from the top corner of the frame and extended vertically downward.

The frame carried 40,500 lb. and the load was held for a few minutes and then dropped very slowly. The cracks were well distributed in the tension zone of the frame and no crack due to diagonal tension was formed. The frame failed by tension in the reinforcement of the beam.

Frame 2—*Inverted U-frame with Columns Hinged at Lower End.*—Nominal span length and total height were each 6 ft. Load was applied at the center of the span of the horizontal member. The location of cracks is shown in Fig. 52. At 8,000 lb. two cracks appeared 2 in. on each side of the center of the top beam and extended upward 2 in. and 3 in., respectively. At 12,000 lb. these cracks had extended vertically 6 in. from the bottom surface of the beam, and a new crack appeared just inside the right-hand corner 10 inches from the center of the beam and extended diagonally toward the load point. At the same load four cracks appeared at the outer shoulders. At 14,000 lb.

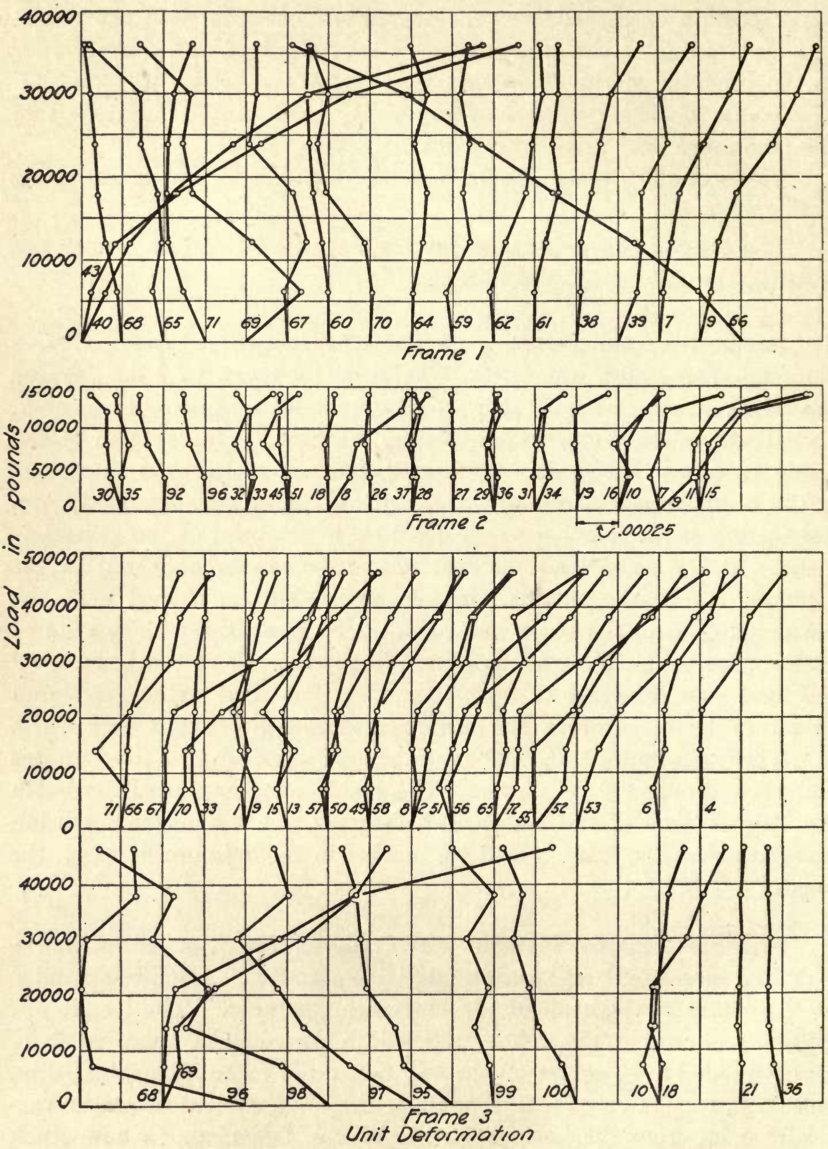
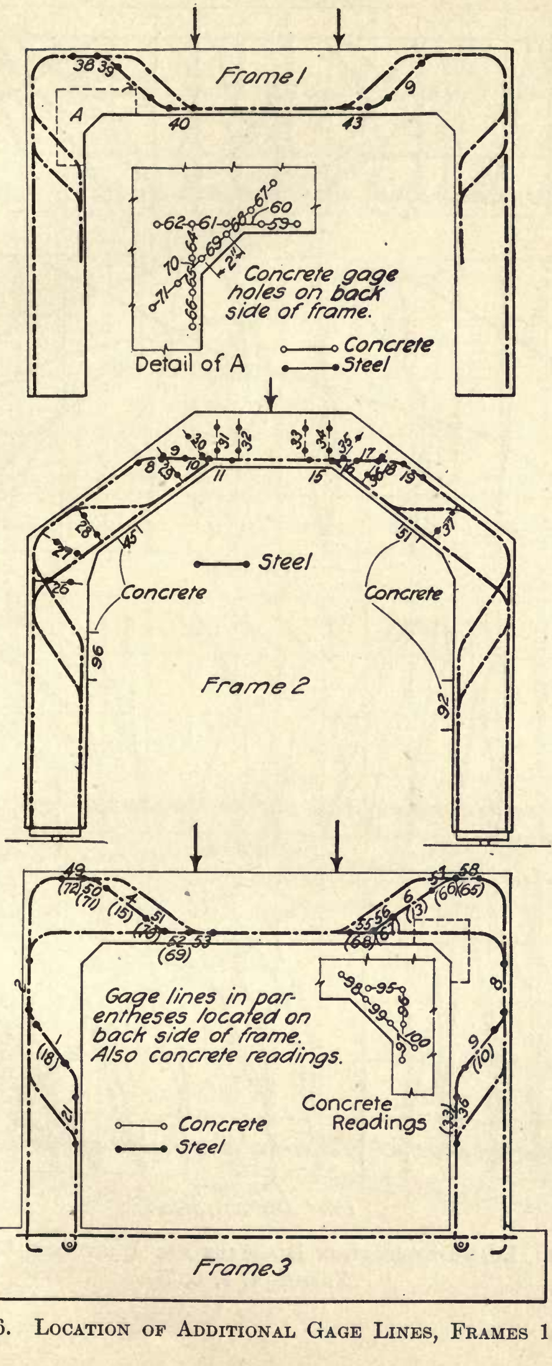


FIG. 45. LOAD-DEFORMATION DIAGRAMS FOR ADDITIONAL GAGE LINES, FRAMES 1, 2, 3



Concrete gage holes on back side of frame.

Gage lines in parentheses located on back side of frame. Also concrete readings.

FIG. 46. LOCATION OF ADDITIONAL GAGE LINES, FRAMES 1, 2, 3

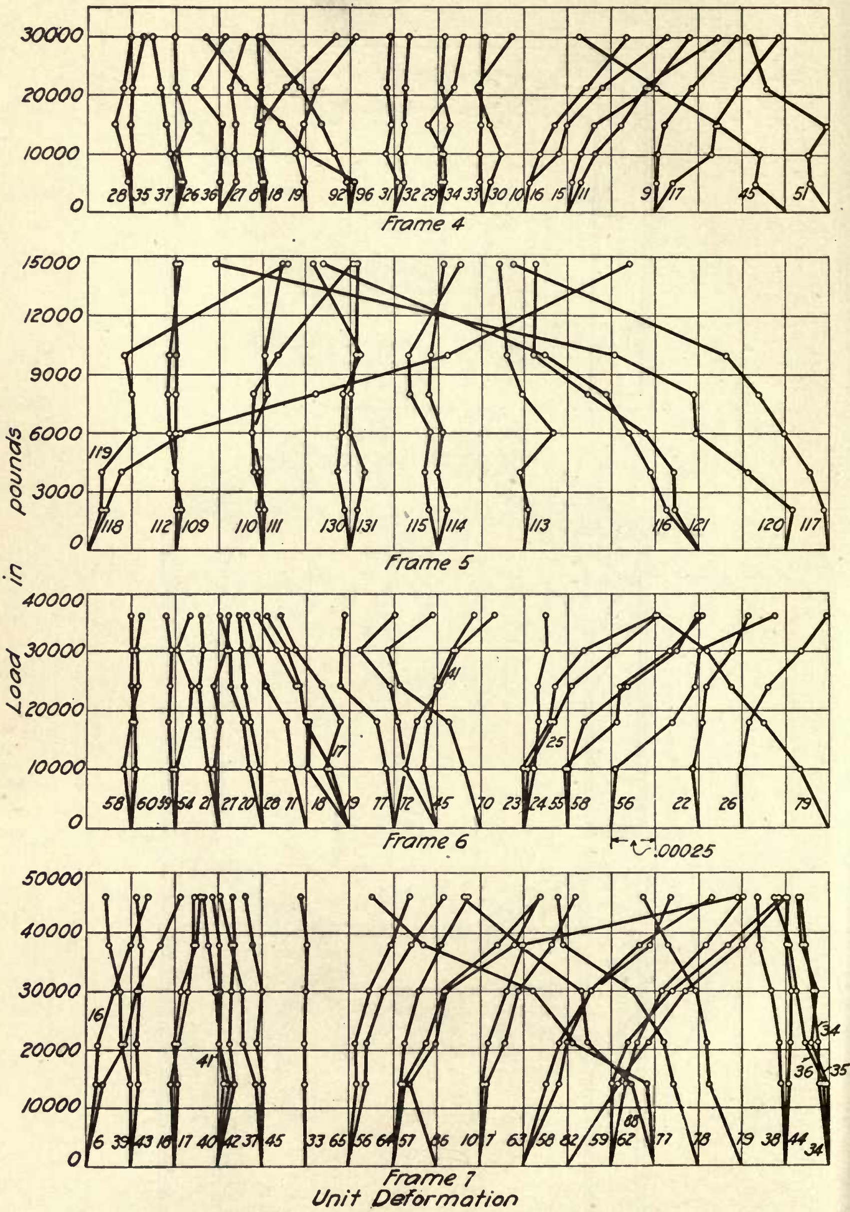


FIG. 47. LOAD-DEFORMATION DIAGRAMS FOR ADDITIONAL GAGE LINES, FRAMES 4, 5, 6, 7

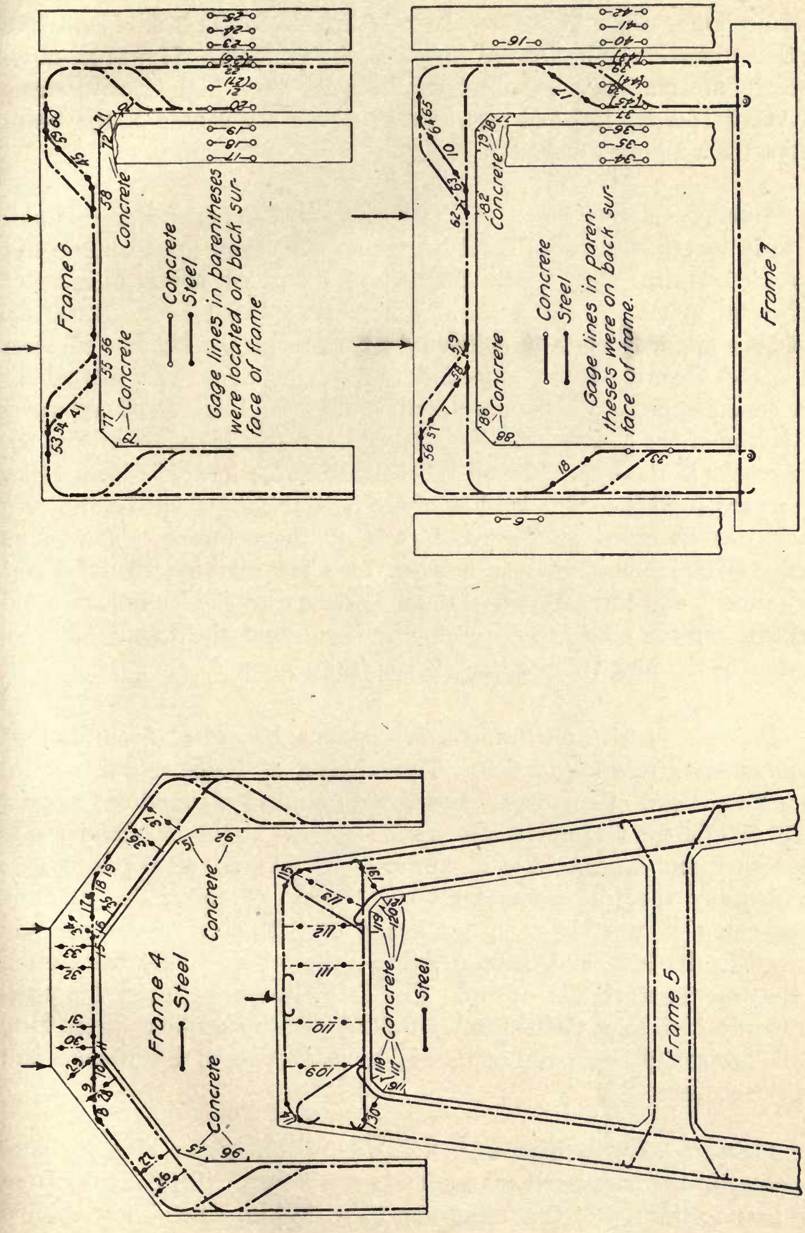

Frame 6
Concrete
Steel
Gage lines in parentheses were located on back surface of frame
Frame 7
Concrete
Steel
Gage lines in parentheses were on back surface of frame.
Frame 4
Steel
Concrete
Frame 5
Concrete
Steel

FIG. 48. LOCATION OF ADDITIONAL GAGE LINES, FRAMES 4, 5, 6, 7

the cracks were extending. Unfortunately at this load the foot of one of the columns slipped outward about ¼ in. due to the lack of sufficient friction to resist the horizontal thrust at the support. However, satisfactory information was obtained because very high tensile stress (32,900 lb. per sq. in.) had been developed at the center of the beam before the slipping occurred.

Frame 3—*Square Frame with Columns Fixed at Lower End.*—Nominal span length was 6 ft. Total height of frame from fixed column end was 4 ft. 11 in. Loads were applied at one-third points of span of horizontal member. No noticeable cracks appeared until 21,000 lb. had been applied, when three cracks appeared at the bottom between loads and several cracks appeared in both columns. At 30,000 lb. new cracks appeared in the beam and columns, and one crack appeared at the top of the beam. The location of the cracks is shown in Fig. 53. The cracks in the upper part of the columns were located within 14 in. downward from the extended line of the bottom face of the beam. No crack was observed at the fixed ends of the columns. The frame carried 60,000 lb. and the load was held for a few minutes, then dropped very slowly, and there appeared to be no danger of sudden failure. No diagonal tension crack appeared in the beam, and the frame failed by tension in the longitudinal steel of the beam.

Frame 4—*Inverted U-frame with Columns Hinged at Lower End.*—Nominal span length was 6 ft. Total height of frame was 6 ft. 3 in. from hinged end of columns. Loads were applied at one-third points. At 10,000 lb. the first noticeable crack appeared at the left-hand inside top corner, and extended 2½ in. upward. Cracks are shown in Fig. 54. Accidentally the frame was built slightly out of form, the columns being out of plumb 1¼ in. in the height of 4 ft., and more stress was thrown to the left-hand column than to the other. The distribution of the cracks shows this clearly. The frame, however, carried a comparatively high load (50,000 lb.). Failure was by tension in the steel in the horizontal beam and at the rigid joint between the columns and the sloped beam.

Frame 5—*Trestle Bent with Tie*—(*A-frame*).—Span length center to center at the supported column ends was 6 feet. Total height from the base to the top of the frame was 10 ft. 1½ in. Load was applied at the center of the top beam. The cross-section of the top beam

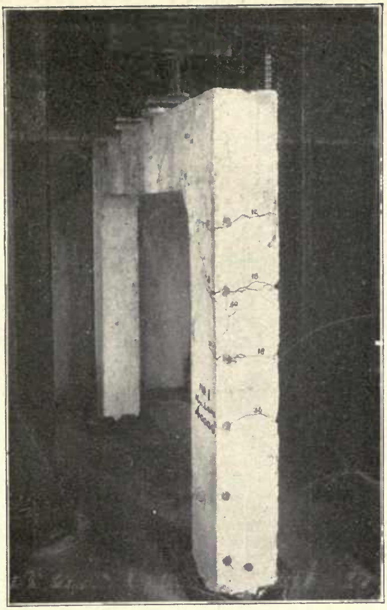


Fig. 49. View of Frame 1 in Testing Machine

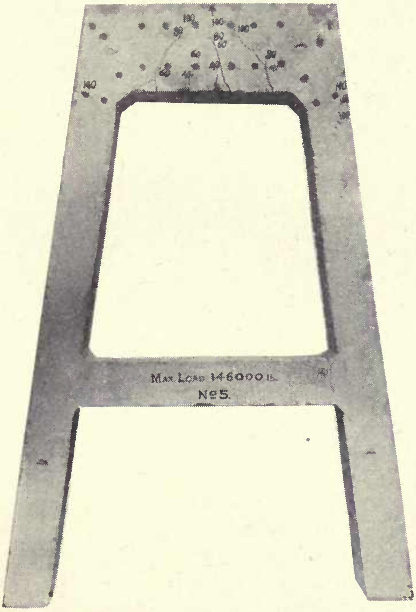


Fig. 50. View of Frame 5 after Test

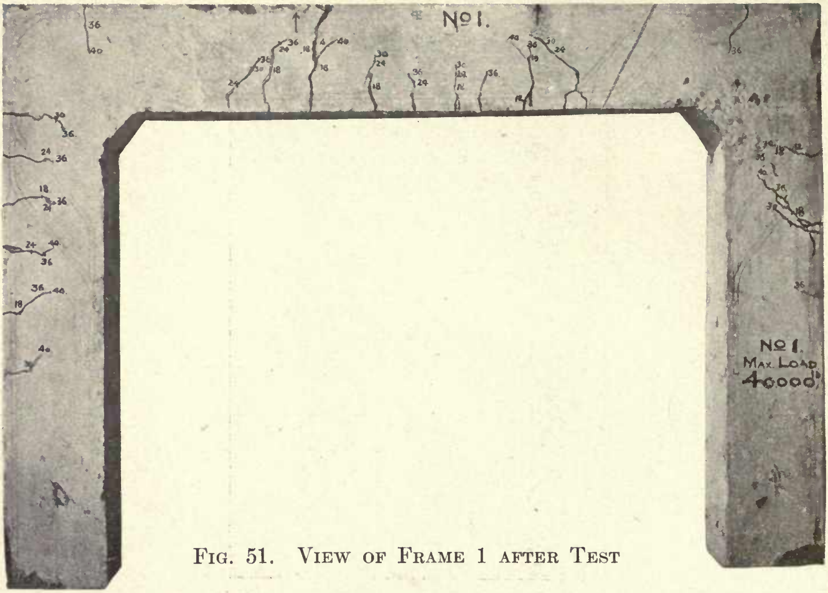


FIG. 51. VIEW OF FRAME 1 AFTER TEST

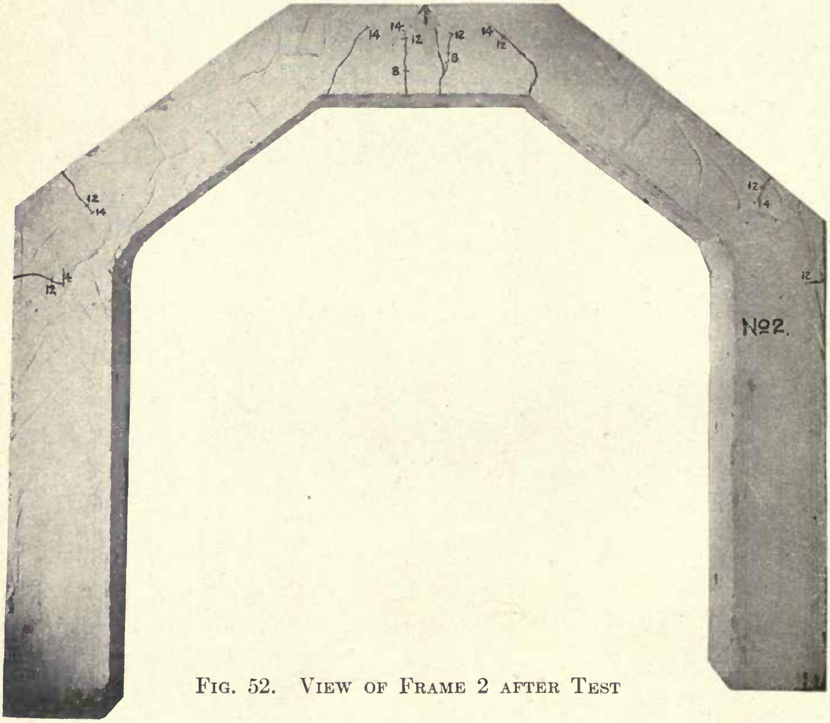


FIG. 52. VIEW OF FRAME 2 AFTER TEST

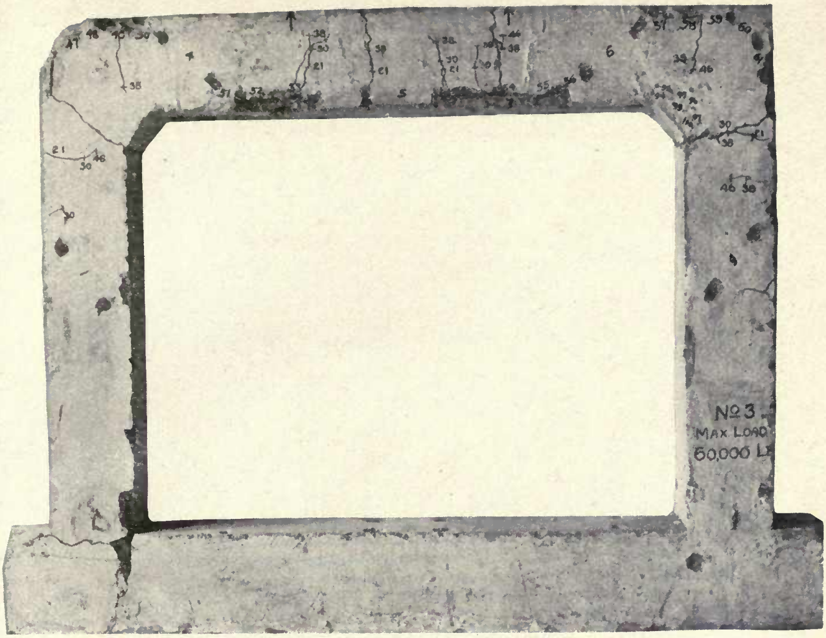


FIG. 53. VIEW OF FRAME 3 AFTER TEST

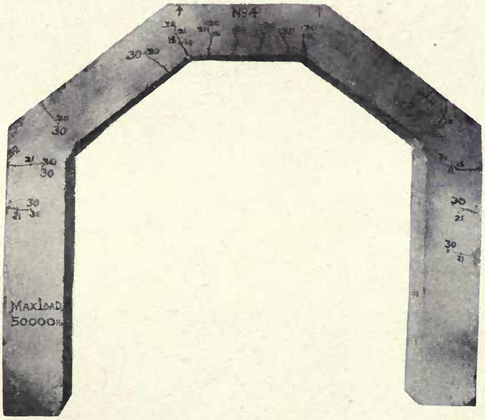


FIG. 54. VIEW OF FRAME 4 AFTER TEST

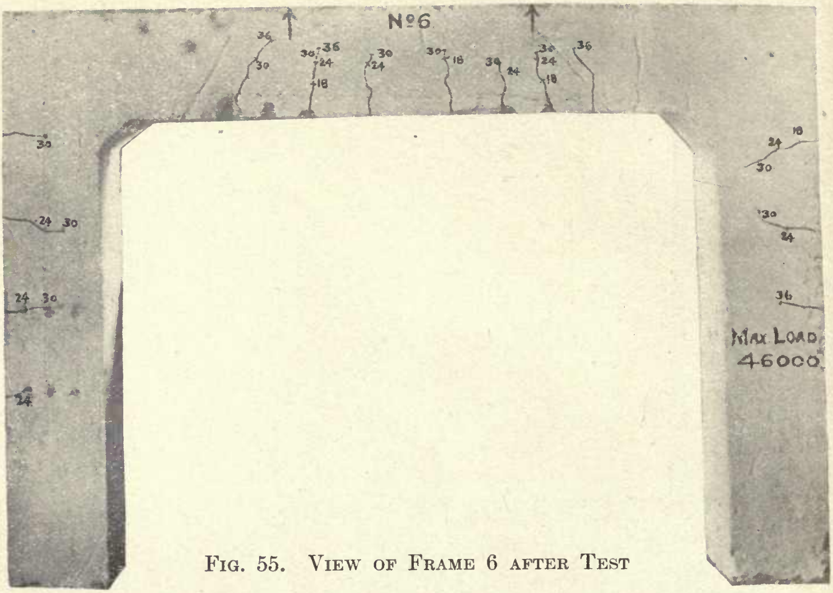


Fig. 55. View of Frame 6 after Test

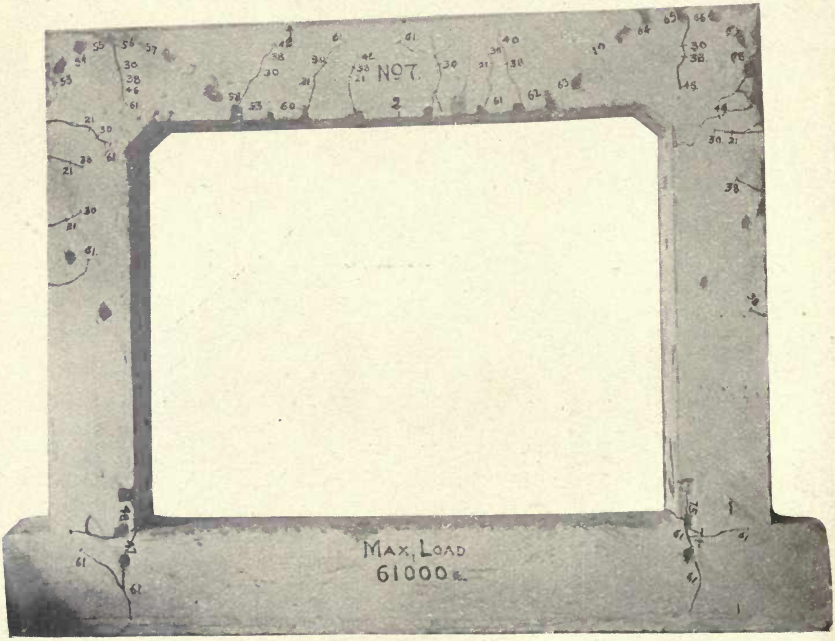


Fig. 56. View of Frame 7 after Test

was 8½ by 16 in. and the column section 8½ by 8½ in. At 40,000 lb. the first two noticeable cracks appeared. These were under the load points of the beam. For location of cracks see Fig. 50. The outer cracks of the beam finally extended diagonally nearly to the load point. At 100,000 lb. the first cracks appeared in the column at the outside near its juncture with the beam. At 140,000 lb. load a crack suddenly occurred at the right-hand rigid joint between the tie and the column with a breaking sound. The frame carried 146,000 lb. and the load gradually dropped. The maximum load was controlled by the failure of the top beam which failed by tension in the reinforcement.

Frame 6—*Square Frame with Columns Hinged at Lower Ends. Same as Frame* 1.—Load was applied at one-third points of span of horizontal member. Fig. 55, a view of the frame after the test, shows the appearance of the cracks. At 18,000 lb. four cracks appeared, two of them under the load points, one near the center of the beam, and one at the upper part of the right-hand column. At 24,000 lb. the cracks extended further and additional cracks appeared in the beam and columns at regular intervals. At 30,000 and 36,000 lb. new cracks appeared in the beam where the longitudinal bars were bent up and these cracks ran diagonally almost to the load points. No crack appeared on the top side of the beam. The frame carried 46,000 lb. and the load then dropped slowly. Failure was by tension in the longitudinal steel in the beam.

Frame 7—*Square Frame with Columns Fixed at Lower Ends. Same as Frame* 3.—Load was applied at one-third points of the span of the horizontal member. The location of the cracks is shown in Fig. 56. The first noticeable cracks appeared at 21,000 lb., three in the beam and three in the columns.

At 30,000 lb., the cracks had extended further and two cracks due to the negative bending moment appeared at the ends 8 in. from the outside face of the columns. At the same load three cracks formed in the bottom half of the beam. As the load increased, the crack located on the outside of the left-hand load point extended diagonally almost to the load point, and the cracks at both ends of the beam extended vertically downward nearly to the bottom side of the beam. The ultimate load carried by the frame was 61,000 lb. At this load sudden failure took place at both inside corners of the lower ends of the columns and the cracks extended horizontally and vertically almost through the concrete base and almost through the columns. This fact shows that considerable positive bending was developed there. The frame also failed by tension in the longitudinal steel of the top beam.

Frame 8—Frame with Three Spans.—Span lengths were 4 ft. 8 in. center to center. Total height of frame was 6 ft. 7¼ in. The base of the frame was 15 ft. 4 in. in length, while the length of the base of the testing machine was 10 ft. 6 in. Consequently the ends of the frame projected beyond the base of the testing machine upon which the frame was bedded in plaster of paris. To observe the upward deflection of the ends of the frame under test an Ames dial was attached at each end of the frame. The movements of both ends were observed as the load increased. The maximum movements (upward) were observed at 60,000 lb. and the amounts were 1/263 in. at the east end and 1/300 in. at the west end. Therefore the steel stress in the beam of the side span may have been slightly modified by the movement, but the structure as a whole probably was not appreciably affected. Uniform load was applied to the upper horizontal member of the middle span. Fig. 57 shows the appearance of the frame after the test with the location of the cracks. No cracks were observed until the load had reached 45,000 lb., when three cracks appeared in the middle span, and one in each outer span on the top side of the horizontal member near the intermediate columns. The former are due to the positive bending moment and the latter are due to the negative moment. These cracks were located symmetrically and they extended vertically about 6 in. At 60,000 lb. they extended deeper. The frame was subjected to the load of 60,000 lb. over 20 hours, but the fall in the applied load was only 300 lb.

At 75,000 lb. several new cracks appeared at both ends of the middle span and also in the upper part of the intermediate columns. At this load the reinforcement at the bottom of the middle beam was stressed in tension beyond the elastic limit of the steel. The frame, however, carried an increasing load in good condition and the highest load was 134,000 lb. At this load the crack at the center of the middle span had opened considerably and the steel at this place had scaled, indicating failure by tension in the steel. At the same time the concrete at the top of the intermediate columns had crushed. Also the concrete base was cracked at the bottom end of the right-hand intermediate column. It is noted that the stresses in the outside columns were very low, even at the maximum load.

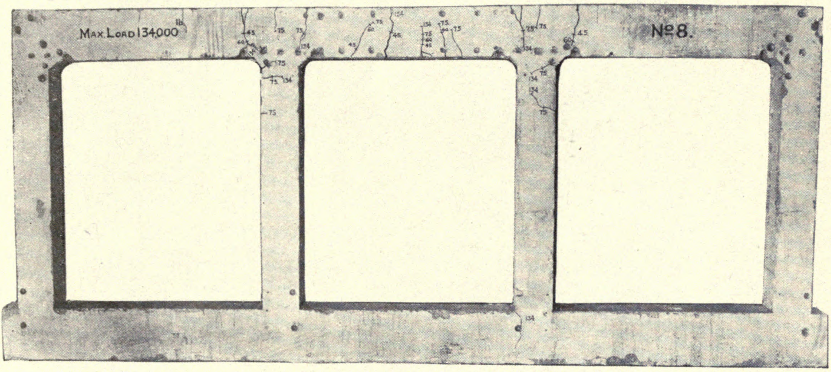


FIG. 57. VIEW OF FRAME 8 AFTER TEST

23. *Conditions Considered in the Comparison of Tests with Analysis.*—In the comparison of test results with analysis, the following modifying conditions have been taken into consideration:

(1) The quality of the concrete was not uniform over the cross-section of the member. The frames were made in a horizontal position on the floor of the laboratory, and the concrete on the rear side of the frame (the bottom side for the position in which the frame was made) seems to have been richer than that on the front side. The concrete on the rear side was stiffer and stronger than that on the front side and the distribution of steel stresses seem to have been modified by this fact. There was more stress in the steel at the rear side of the member than in the steel at the front side. It appears reasonable to take the average value of the observed stresses in the part in question for the purpose of comparison.

(2) The steel stresses are greatly modified by the presence of tension in the concrete for the low loads. Therefore in comparing the computed stresses with the observed stresses, two cases must be considered, one in which the concrete is considered to take tension and the other in which the concrete is considered to be broken in tension.

(3) The cross-sections of the test frames were designedly made larger in proportion to the span than would commonly be used in practice. In most of the test pieces, the column width occupied nearly one-seventh of the nominal span (distance center to center of the columns). In addition to this the corner at the juncture of the beam and the column was provided with a fillet. Under these conditions the bending moment at the center of the beam will be less than that calculated on the basis of the nominal span length—the difference being greater than that occurring with the dimensions found in practice. In the computations for the horizontal reactions the nominal span and height of the frames (distances center to center) were used. In finding the numerical values of the bending moment in the beam, and also of those in columns having fixed ends, the horizontal reactions computed as described previously were used, but when the equations involved further use of span lengths the nominal span length was replaced by the clear length of the span.

(4) The design of Frames 1 to 7 was such as to cause high stresses in columns and beams at about the same time. In Frame 8 the presence

of the members of unloaded panels caused the moments in the columns to be much smaller than in the loaded beam. The base also offered so much restraint as to give the column ends almost a fixed condition. Consequently loads which would cause high stresses in the beams would not produce cracks even in the intermediate columns although the moment there would be greater than in the outer columns. It will be seen then that for Frame 8 there are two cases to be considered in comparing the experimental results with the analyses, one in which tensile strength in the concrete is considered in all members and the other in which the beams and the intermediate columns are cracked on the tension side. The bending moment in the outside column is very small, and there is no chance for tension cracks. The moments in the intermediate columns and in the beams of the side spans are also small except at the extreme end, and only one crack appeared in this member. Therefore in the calculation of the moment of inertia of the cross-section for the second case it is not correct to neglect entirely the tensile strength of the concrete in these two members. A probable value for the moment of inertia of these members will be an average between that obtained by using the full cross-section and a section which neglects the part outside the tension rods. This assumption was made in the numerical computation of the moments and stresses for Frame 8.

In making the computations of stress in the concrete of the frames a constant modulus of elasticity was used, that is, a straight line stress-deformation relation was assumed. The selection of a proper value of the modulus of elasticity was somewhat dependent upon a knowledge of the qualities of the concrete of the various frames and a comparison of the behavior of the frames with that of the corresponding control cylinders. Naturally the modulus of elasticity used was generally less than the initial modulus. The values of modulus of elasticity of the concrete used in the computations of stress are given in Table 11.

24. *Comparison of Test Results with Analyses.*—The observed stresses, i. e., those obtained from the observed deformations by the process already described, have been plotted in Fig. 36 to 43. The light full line represents stress in the steel at the nearer side of the specimen; the dashed line the stress in the farther side of the specimen. The dotted line represents stress in the concrete, usually at the median plane. In general, the stresses have been plotted from the central longitudinal axis of the member in which they were observed. Because of the possibility of confusion resulting from this method an exception

was made in Frames 2, 5, and 8, where the stresses have been plotted from the line showing the location of the reinforcing bar. Tension in steel and compression in concrete when measured at or near the inner surface of a member have been plotted in, that is, toward the central part of the frame as a whole; tension in steel and compression in concrete when measured at or near the outer surface of a member have been plotted out. In only a few cases have tension in concrete and compression in steel been plotted. Both have been plotted out if measured on the inner surface of a member and in if measured on the outer surface. With very few exceptions, consequently, steel stresses represent tension regardless of whether in the diagrams they appear as positive or as negative with reference to the coördinate axes. Plotted concrete stresses, likewise, represent compression. The point at which the stress line crosses the coördinate axis represents the position of the point of inflection and not a change in the sign of the stress. The heavy full line represents the computed stresses for both steel and concrete which have been calculated for various points by means of the formulas given in preceding pages.

In Fig. 45 and 47 are given load-deformation curves for gage lines not represented in the diagrams of Fig. 36 to 43. The location of the gage lines is shown in Fig. 46 and 48.

In Table 14 are given values of both observed and computed stresses at three points for several loads for all the frames. Where I precedes the computed stress the calculation considered the tensile strength of the concrete; where II precedes it and where neither I nor II is given, the computed stress is based on the assumption of no tensile strength in the concrete.

A study of the tables and diagrams seems to justify the following statements:

(1) The experimental and the computed values of steel stress at the center of the loaded top beam are in fair agreement for each kind of frame tested except in a few instances in which the load was comparatively low or extremely high.

(2) The experimental and the computed values of the steel stress in the columns are also in fair agreement, but the maximum difference between experimental and theoretical values is higher than in the beams, owing to the fact that the direct stress is not equally distributed over the cross-section of the column.

TABLE 14

COMPUTED AND OBSERVED STRESSES IN FRAMES

Computed Values are given in Roman Type; Observed Stresses in Italics; f_s is Unit Stress in Steel; f_c is Unit Stress in Concrete.

Frame and Load	A		B		C	
	f_s	f_c	f_s	f_c	f_s	f_c
Frame 1						
18 000	14 800	710	10 800	.....	16 900	840
	16 100	*1 140*	*8 900*	.....	*17 900*	*1 240*
30 000	24 600	1 180	18 000	.....	28 200	1 400
	25 300	*2 560*	*17 400*	.....	*28 600*	*2 500*
36 000	29 500	1 420	21 600	.....	33 800	.....
	31 900	.....	*21 300*	.....	*36 000*	.....
Frame 2						
12 000 I	1 800	490	......	.....	......	.....
II	11 600	710	11 200	800	21 000	.....
	5 000	*690*	*11 600*	*730*	*21 100*	.....
Frame 3						
30 000 I	600	480	......	.....	......	.....
II	.....	.....	13 200	1 080	22 800	1 350
	1 400	*250*	*11 900*	*1 520*	*18 600*	*820*
38 000 I	900	610	......	.....	......	.....
II	.....	.....	16 700	1 370	28 900	1 710
	2 000	*340*	*16 700*	*1 840*	*25 700*	*1 190*
46 000 I	1 100	740	......	.....	......	.....
II	.....	.....	20 200	1 660	35 000	2 070
	2 800	*400*	*22 300*	*1 850*	*34 300*	*1 500*
Frame 4						
10 000 I	1 500	410	2 000	340	......	.....
II	8 500	540	8 700	630	8 700	630
	3 400	*590*	*3 500*	*680*	*7 200*	*730*
21 000 I	3 200	860	......	.....	......	.....
II	17 800	1 130	18 300	1 320	18 300	1 320
	11 700	*1 550*	*16 300*	*1 670*	*19 300*	*1 540*
30 000	25 500	1 620	26 100	1 890	26 100	1 890
	21 400	*2 350*	*23 700*	.....	*29 900*	*2 550*
Frame 5						
40 000 I	— 1 900	— 3 000[1]	— 900	— 3 900[1]	.....	.....
II	.....	......	......	......	12 000	.....
	— 2 000	*— 3 700*[1]	*+ 1 300*	*— 2 200*[1]	*10 100*	.....
80 000 I	— 3 700	— 5 900[1]	— 1 900	— 7 800[1]	......	.....
II	.....	......	+15 200	— 5 300[1]	24 000	.....
	— 2 400	*— 7 300*[1]	*+ 4 500*	*— 8 500*[1]	*24 800*	.....
100 000 I	— 4 700	— 7 400[1]	— 2 400	— 9 700[1]	......	.....
II	.....	......	+19 000	— 6 600[1]	30 000	.....
	— 2 000	*— 9 400*[1]	*+ 9 300*	*—10 900*[1]	*33 500*	.....

TABLE 14—(CONTINUED)

COMPUTED AND OBSERVED STRESSES IN FRAMES

Computed Values are given in Roman Type; Observed Stresses in Italics; f_s is Unit Stress in Steel; f_c is Unit Stress in Concrete.

Frame and Load	A		B		C	
	f_s	f_c	f_s	f_c	f_s	f_c
Frame 6						
18 000 I	1 800	540	2 200	600	4 000	760
I	.....	.....	......	.....	19 600[2]	1 170[2]
II	.....	.....	10 800	830	9 900	840
	3 700	*900*	*6 900*	*1 090*	*9 300*	*1 080*
30 000 I	3 000	900	......	.....	31 800[2]	1 950[2]
II	24 300	1 320	18 000	1 380	26 500	1 680
	14 800	*1 670*	*17 700*	*2 430*	*21 900*	*2 050*
36 000 I	.....	.....	......	.....	38 200[2]	2 340[2]
II	29 100	1 930	21 600	1 660	32 900	2 020
	23 600	*2 460*	*22 100*	.....	*28 100*	*2 750*
Frame 7						
21 000 I	900	3 700[1]	2 700	610	......	.....
II	.....	.....	9 000	800	16 000	1 000
	— *100*	*3 400*[1]	*4 900*	*1 350*	*13 100*	*1 530*
38 000 I	1 600	6 800[1]	4 900	1 100	......	.....
II	.....	.....	16 300	1 440	28 900	1 830
	400	*6 000*[1]	*12 100*	....[3]	*27 500*	*2 900*
46 000 I	1 900	8 200[1]	......	.....	......	.....
II	.....	.....	19 800	1 750	35 000	2 210
	700	*7 700*[1]	*16 500*	....[3]	*37 000*	....[3]
Frame 8						
30 000 I	8 900	850	5 000	6 500[1]	1 800	1 800[1]
	7 000	*1 490*	*1 300*	*5 100*[1]	*2 100*	*3 600*[1]
60 000 I	27 700[2]	1 700[2]	9 900	13 000[1]	3 600	3 600[1]
II	......	.....	.....	......	18 300	2 800[1]
	24 500	*2 540*	*7 200*	*13 100*[1]	*12 800*	*6 200*[1]
75 000 I	......	.....	12 400	16 200[1]	......	.....
II	38 500	2 360	......	......	23 000	3 500[1]
	37 500	*2 750*	*11 500*	*17 700*[1]	*19 100*	*6 700*[1]

---

[1] Compressive stress in steel.
[2] Concrete in beam under load considered as broken in tension.
[3] Very high stress.

(3) The observed compressive stresses in the concrete at the low loads which developed a unit-stress up to about 800 lb. per sq. in. agree reasonably well with the computed values though in most of these cases the observed concrete stresses were somewhat higher than the computed stresses. In some instances the discrepancies ran up to 50 per cent, and for higher stresses the discrepancies were frequently even greater. Undoubtedly these differences are partly due to the fact that the modulus of elasticity used does not represent correctly the modulus of elasticity of the concrete in the frames. Other matters difficult of explanation probably cause further discrepancies.

25. *Effect of End Condition of Column on Results.*—The secondary stresses which would be expected as a result of the friction in the bearings at the free ends of the columns in Frames 1, 2, 4, and 6 seem to have been very small and may be neglected without appreciable error.

The concrete bases used for Frames 3, 7, and 8 to secure the fixity of the column ends were, of course, not entirely rigid, and a slight bending in the base due to a load may be expected to have an influence on the bending in the other members. Deformation readings at the middle point of the base were taken at each increase in the load. The results of these observations showed practically no bending stress for all loads except the ultimate load.

26. *Distribution of Stress over the Cross-Section.*—In the observations it was found that stress in the steel on bars near a front corner of a member differed from that on bars near a back corner of the member, the front of the member being the top side of the frame as poured and the back side being the bottom. In Table 15 are given stresses in bars at front and back at two places on the frame for one load generally near the maximum. It is seen that generally the stress in a bar near the front of the member (top of the member as poured) is less than that in a bar near the back (bottom of the member as poured). To investigate the distribution further, special measurements were made in the columns of Frames 6 and 7, the gage lines being placed where bending was not sufficiently large to produce tension cracks in the concrete. The gage lines were located on the four faces of the column, and the observations were made at each load. The front outer corner developed the lowest tensile stress and the front inner corner the highest compressive stress. The back outer corner developed the highest tensile

Table 15

Stresses in Bars at Front and Back of Members of Test Frames

	At Center of Span				At Upper Part of Column			
	Observed Stress in Steel				Observed Stress in Steel			
Frame No.	Back Side	Front Side	Difference	Per cent	Back Side	Front Side	Difference	Per cent
1	22 200	27 100	— 4 900	22.3	28 600	35 200	— 6 600	23.1
2	32 900	27 400	+ 5 500	16.7	11 300	9 500	+ 1 800	15.9
3	39 500	29 000	+10 500	26.6	25 500	18 900	+ 6 600	25.9
4	34 400	24 000	+10 400	30.2	31 000	27 600	+ 3 400	11.0
5	36 400	30 600	+ 5 800	16.4	35 600	22 700	+12 900	36.2
6	29 800	26 400	+ 3 400	11.4	23 500	19 300	+ 4 200	17.9
7	33 500	21 500	+12 500	35.8	22 300	10 500	+11 800	52.9
8	30 900	18 400	+12 500	40.4	21 400	15 200	+ 6 200	29.0

stress and the back inner corner the lowest compressive stress. The distribution was not much altered by the increase of load.

The observations indicate a lateral bending and twisting of the frame. It seems probable that the main source of the difference in stresses from front to back was the lack of homogeneity of the concrete, that at the bottom of the member as poured being stronger and stiffer than that at the top. A difference in stiffness would at least partially account for the phenomena.

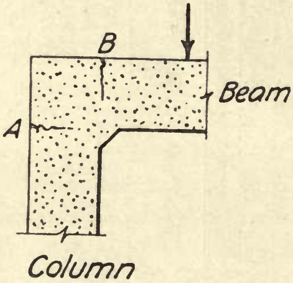


Fig. 58. Connection of Beam and Column Showing Typical Location of First Cracks

27. *Position of Point of Inflection in Columns.*—The position of the point of inflection in a member of a structure which is subject to flexure is an important element for use in designing the frame. To determine from observed deformations the position of the point of inflection for a member, it is necessary to separate the deformation into that caused by direct stress and that due to flexure of the member. The deformation in the columns of Frames 3 and 7 were thus separated, a straight line stress-deformation relation being assumed, and the position of the point of inflection found. The position of the point of inflection in the columns of these frames changed very little during the progress of the loading. For these frames the point of inflection was found to be almost exactly at one-third the height of the column, as is indicated by the analysis.

28. *Continuity of the Composing Members of a Frame.*—In the tests of the frames there was no sign of discontinuity of members whatever. It is apparent from the action of the frames and from the stresses

observed that the stresses and therefore the moments were well transmitted from member to member by the connection. From the results it is felt that there is every reason to have confidence in the rigidity of connections in frames that are properly designed.

In the frames free to turn at the lower column ends there was a tendency for a crack to form near the juncture at A (Fig. 58) at a lower load than that at which a crack appeared at B. In the frames with rigid connection at the lower column ends, the crack at B appeared at nearly the same time as that at A.

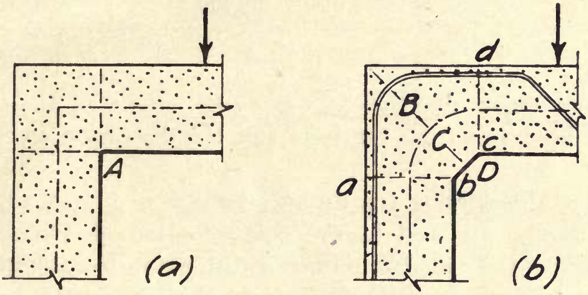


Fig. 59. Rectangular Joint with and without Fillet

29. *Stresses at Corners.*—In the design of a frame a square corner such as that shown at A in Fig. 59 (a) should be avoided for all connections, for it is well known that theoretically in resisting bending the material at the corner would develop excessively high stresses. It is therefore common to design such corners with fillets as shown in Fig. 59 (b). No attempt will be made to compute the stresses at the fillets. The observed deformations at gage lines in the neighborhood of the fillets and within the space occupied by the intersection of the two members are plotted in Fig. 45 and 47. These deformations are of interest. Some of these values have been converted into concrete stresses by the use of the moduli of elasticity of the concrete already assumed and are given in Table 16.

TABLE 16

OBSERVED STRESSES AT SHARP CORNERS IN FRAMES 1, 3 AND 6

Stress in lb. per sq. in.

Load lb.	Frame 1 Gage Line 69	Frame 3 Gage Line 81	Frame 6 Gage Line 71
7 000	....	300	....
10 000	....	....	310
12 000	80	....	....
14 000	....	720	....
18 000	670	....	450
21 000	....	760	....
24 000	810	....	900
30 000	....	1 460	1 100
36 000	1 410	....	1 320
38 000	....	2 120	....

30. *Conclusions and General Comments.*—Some of the conclusions which may be drawn from the tests and the discussion are as follows:

(1) Considering the errors involved in the measurement of the deformations and in the determination of the modulus of elasticity of the concrete, as well as those due to assumptions with reference to the distribution of stresses across the section and over the gage length, the results presented indicate a fair agreement between analyses and tests and justify the conclusion that the formulas given in the bulletin for statically indeterminate stresses as applied to reinforced concrete structures will give values for stresses in the members well within the limit of accuracy required in design.

(2) The elastic action of the frames under external load and the manner of stress distribution along the members of the frame agree fairly well with the analyses given.

(3) The location of the point of inflection in the members of the frames under load agrees closely with the location found by analyses.

(4) If a frame is carefully designed and well reinforced, there need be no anxiety as to the rigidity of a joint. Effective continuity of members has been found in the tests.

(5) No sudden failure took place in the frames tested. The increase in the deflection was uniform, indicating as great reliability for reinforced concrete frames as for steel structures.

(6) The load at which the first fine crack appears near the juncture of members is increased by fixing the lower column ends of a frame. This is obviously due to the increase in horizontal thrust at the lower column end over that developed when the lower end is free to turn.

(7) At sharp inside corners, high compressive stresses were developed in the concrete due to so-called curved beam action and in several cases local failure occurred by the crushing of the concrete at these corners under high loads.

(8) A slight deviation of the axis of vertical members from a vertical line, that is to say, a slight "out-of-form" of the vertical columns, produced an appreciable variation in the stress distribution in the frame.

(9) Owing to the existence of a horizontal thrust (which varies from $\frac{1}{8}P$ to $\frac{1}{18}P$ in most common cases of simple frames) at the ends of a vertical or inclined member, it is advisable to incline the member slightly toward the direction of the reaction at the end. Such arrangement will greatly reduce the bending stress in the member. If this arrangement is not practicable, a slight increase in the top width of a vertical member and a slight decrease in its bottom width, brought about by inclining the inner surface and making the outer surface vertical will add materially to the rigidity of a frame without a proportional increase in the amount of material used.

(10) For a frame having an inclined column, it may be possible to select the form of frame in such a way that the column will take no bending stress throughout its length.

(11) Due attention should be paid to the rigid joint of a tie member to insure the stiff connection with a main member. A marked tendency to cause a sudden breaking of such a joint accompanied an increase of bending moment in a main member.

(12) The use of a footing rigidly connected to the lower end of a vertical member is advisable, for it will reduce the bending moment at the juncture of the vertical and inclined members. A frame having such a footing is solvable analytically, since it approaches the case halfway between that of the hinged end and that of the fixed end of the vertical member, provided the foundation is sufficiently unyielding. A little consideration is needed to provide proper reinforcement at the juncture of the column and the footing.

(13) The formulas derived by analysis may be applied to a variety of forms of frames and are of wide applicability.

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